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The Analysis of Masonry Architecture: A Historical Approach

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To the memory of Professor Henry J. Cowan

Abstract: Traditional unreinforced masonry architecture has disappeared from new building activity in the western world. Nevertheless, the architectural heritage of masonry must be preserved, and this involves structural analysis. The classical theory of structures does not apply well to such heterogeneous structures with unknown boundary conditions. Nevertheless, there exists a theory of masonry structures based in simple assumptions about the material: good compressive strength, almost no tensile strength and a constructive care to avoid sliding failure. The theory was born at the end of the 17th Century, developed during the 18th and was applied in the 19th Century. It was abandoned and eventually forgotten at the beginning of the 20th Century. After half a century, in the 1960s, Heyman incorporated the old theory within the frame of modern limit analysis with its implicit treasure of critical observation and experience. The safe theorem permits using equilibrium equations and simple material statements cited. No affirmation about boundary conditions, impossible to know and essentially changing, is made (other than the usual about strength and small displacements). In the first part of the paper, an outline of the old theory is summarised and discussed. In the second part, the main ideas and concepts of limit analysis of masonry structures are discussed.

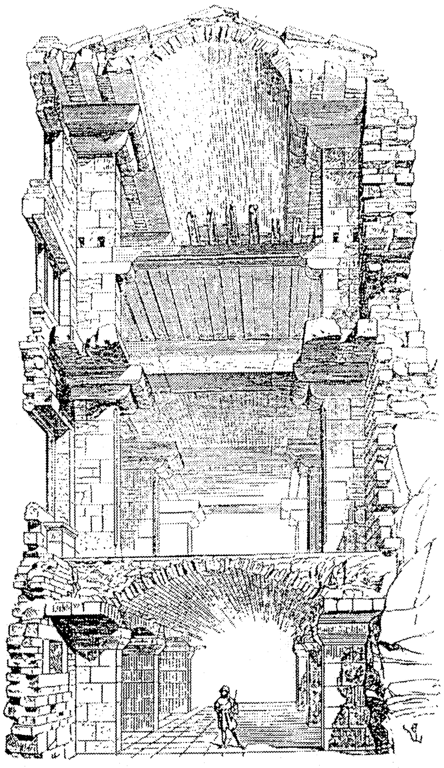
Keywords: Arches, Architectural heritage, Architectural history, Building materials, Domes, Elastic analysis, Finite element method, Graphical equilibrium analysis, History, Limit analysis, Masonry architecture, Masonry structures, Safety, Stability, Statics, Structural behaviour, Theory of structures, Vaults

Introduction

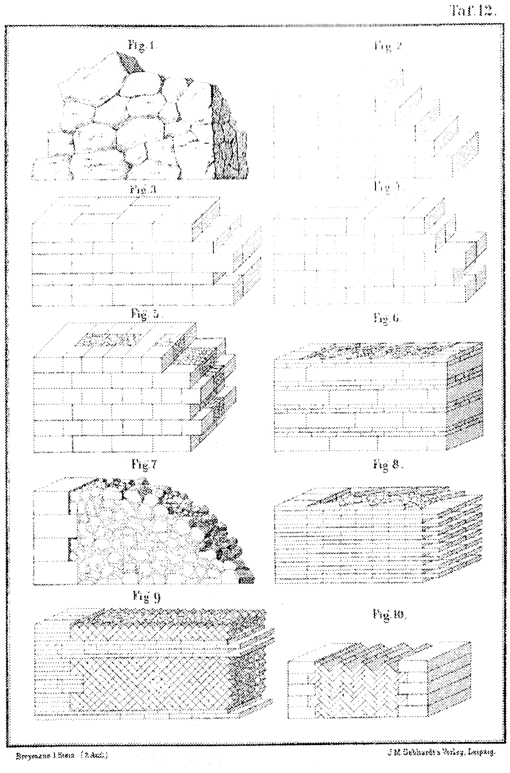
Masonry was the main building material in the western world until the beginning of the 20th Century. Masonry vaults are no longer built, but we have to preserve our architectural heritage. In many cases, it is a matter of cosmetic maintenance. However, sometimes, the building presents, or seems to present, structural problems. It may happen, also, that the loads have increased and the safety of the structure should be checked for the new use. In these last situations, a structural analysis is needed.

The problem is that masonry structures are essentially different from modern structures of steel, reinforced concrete or laminated wood. The usual structural theory of framed, trussed or shell structures made of reinforced concrete or steel is of no use to study masonry architecture. In Figure 1a, a constructive section of a medieval building is represented. In the first place, no such linear elements as columns and beams are seen. The structural elements are two- or three-dimensional, but not linear, as in frame or truss theories, or thin as in the usual shell theory.

Let us consider now the material. Though from the outside, regular ashlar masonry would be seen, the internal structure is, in fact, much more irregular and complex. The wall, for example, consists of two skins of ashlar masonry and a nucleus made of rubble with some kind of mortar. The masonry material is, in itself, a structure. How to apply, then, the usual material assumptions: continuity, isotropy, definite elastic or non-elastic properties? Figure 1b shows a few of the usual masonry types. If we try to define, for example, an elastic modulus: Where? At the outer skin, in the inside skin, at the nucleus? Besides, it is a fact that masonry buildings present cracks, and these cracks run often through the whole thickness of walls and vaults. They may have been closed at the surface (by some plasterwork), but the crack remain inside, hidden and unknown. Finally, speaking of boundary conditions, it is well known that the foundations of most masonry buildings are superficial and present noticeable settlements: they are far from the rigid foundations of the structural textbooks. They are unknown, and essentially unknowable, as slight changes of the soil conditions, the sudden action of loads (e.g., storms or earthquakes) could alter the response to the loads.



(a)



(b)

Figure 1: *Cross-section of a medieval building showing the internal structure (Viollet-le-Duc, 1854-1868), (b) different types of masonry (Warth, 1903).*

The above remarks do not imply that masonry structures cannot be designed or analyzed. In fact, some kind of design was needed to build the Pantheon, Hagia Sophia, and the great Gothic cathedrals. It was not an amateur task, nor a matter of blind chance. There was, indeed, a traditional *scientia* of structures, based mainly in the geometrical study of safety. This pre-scientific theory has been discussed elsewhere (Huerta, 2004), but our concern is, now, how to analyze these buildings within the frame of the scientific theory of structures, based on the laws of statics and the strength of materials. We have briefly discussed the difficulties, even the impossibility, of applying directly the usual assumptions of modern common structural analysis. Another approach is needed.

The Theory of Masonry Structures

Fortunately, there exists a scientific theory of masonry structures. It was born at the end of the 17th Century, developed during the 18th and 19th Centuries, and was forgotten, and eventually disappeared from structural textbooks and curricula in the mid-20th Century. The elastic theory swept away the then so-called old theory of arches and vaults. Indeed, the old theory of vaults was of no use to design steel frames, suspended bridges, lattice iron arches, etc. However, this theory was used successfully for arch and bridge design all through the 19th Century and was applied, again successfully, to the study of masonry buildings, particularly to check the stability of the new neo-Gothic masonry buildings at the end of the 19th and during the 20th Century.

The old masonry arch theory began to be rediscovered in the 1930s when engineers were forced to give concrete answers about the strength capacity of medieval bridges for new loads. Eventually, in the 1960s, the old theory of masonry structures was incorporated within the modern frame of Limit Analysis, mainly thanks to the work of Jacques Heyman of Cambridge (1966 ff). Heyman had participated actively in the developing of the plastic theory of steel frames within the famous team of John Baker in the 1940s and 1950s. He realized, crucially, that masonry was in fact a ductile material. Though the individual stones may be brittle, he saw clearly that the apparent shortcomings (from the classical elastic point of view) of the old theory, where not such, and that the correctness and the success of the old approach could be explained within the broader frame of modern Limit Analysis of masonry structures.

We should use this theoretical frame if we want to understand the behaviour of masonry buildings. However, the coming of computers, the exponential cheapening of calculation costs, the modern multi-purpose Finite Element or Discrete Element packages, with the implicit or explicit promise of being the universal method to be used for any kind of structure, have obscured the actual situation and the potentialities of the new theory of masonry.

Indeed, in the last decade, the number of papers and books concerned with the structural analysis of historical masonry architecture has increased exponentially. The matter which some twenty years ago interested only architects and engineers involved in restoration works has become the subject of numerous scientific and academic papers, of research projects, dissertations, congresses and seminars. The work in instrumentation and analytical aspects has been, also, enormous. All the machinery of the modern computational methods and the latest analytical techniques has been addressed to the study of old masonry buildings, many of them more than a millennia old. However, it does not seem that all this display has led to a better understanding of masonry structures. Quite often masterpieces of architecture are suffering invasive interventions with insertion of reinforced concrete, steel, glass fibre, Kevlar, epoxy, and so on. Almost any building under study is considered in danger and scrutinized in detail and, almost invariably, followed the usual insertion of modern reinforcement of one kind or another. No doubt, sometimes a historical building requires a structural consolidation, but what is remarkable is the extensive use of such modern techniques.

Within a historical perspective, it is a recent phenomenon. Until, say, 1900 buildings were restored and consolidated using the same materials and techniques with which they were built, following the tradition of masonry construction. Old buildings were looked at with confidence as they have withstood many proofs during their lives. There is nothing against the use of new materials and techniques when needed, but it may be that this extensive use is the expression not strictly of the bad state of the buildings, but of the bad state of the understanding of old construction. It often occurs that interventions tend to convert masonry to what it is not: a monolithic structure that can withstand tensions as well as compressions.

The present paper is intended to examine this crucial matter and to open a discussion and reflection in the way structural analysis and consolidation of old buildings is made today. The discussion will be centred in the theory and not in the computational techniques. Being the matter of historical construction, it may be justified to use a historical approach to obtain a more ample view.

In the first part of the paper, an outline of the "old" theory of masonry structures will be given. In the second part, the ideas and concepts of modern Limit Analysis will be exposed, and its main corollary, the *equilibrium approach*, will be discussed.

The Theory of the Arch: An Historical Outline

The main lines of the development of arch theory are well known. The works of Poncelet (1852) and Winkler (1879-80) give a good review of the early theories from the 17th to the mid-19th century. Those theories refer to masonry arches (often called *rigid* in 19th Century engineering manuals). The theory of the elastic arch was developed during the 19th Century and was applied first to iron and wooden arches. After the 1880s, it was applied to any kind of arch. A detailed study of the history of the elastic theory may be found in Mairle (1933) and a good review of the (1941), Timoshenko (1953) and Charlton (1982). Heyman (1972, 1998b) has studied the evolution of masonry arch theory within the frame of limit analysis, and has placed it rigorously within the general frame of the modern theory of structures (Heyman, 1966, 1995a, 2008). Kurrer (1997) covered both the history of rigid (masonry) and elastic theories, and a overall view may be found in his books on the history of structural theory (Kurrer, 2002, 2008). Foce (2002) has contributed another historical review and, more important, has compiled a comprehensive bibliography of the primary sources. Finally, the present author has contributed jointly with Kurrer another historical outline of masonry arch and vault theory (Huerta & Kurrer, 2008).

England: The Equilibration Theory

It was Robert Hooke who, circa 1670 (see Hooke, 1676) posed the problem of the arch (Heyman 1972): What is the ideal form of the arch, and how large is the thrust of the arch against the abutments? He gave the solution in the form of an anagram in 1675 (cf Hooke, 1676), which, deciphered, says: As hangs the flexible line, so but inverted will stand the rigid arch (Figure 2a). The statics of arches and hanging chords is the same. This was Hooke's genial idea. The mathematician Gregory (1697), in an essay on the catenaria, gave the same statement as Hooke and added a fundamental remark: None but the catenary is the figure of a true legitimate arch, or fornix. Moreover, when an arch of any other figure is supported, it is because in its thickness some catenaria is included. Hooke's analysis gave a simple approach to understand and calculate masonry arches, and Gregory's comment freed the form of the arch from that of the catenary. However, Gregory's fundamental statement was ignored.

Following Hooke, English engineers tried to design the arch of the same form of the corresponding hanging chain, to build an equilibrated arch. The matter was tackled mathematically by many mathematicians and engineers in the second half of the 18th Century, for example, by Emerson (1754) and Hutton (1772). The arch of uniform thickness, which sustains only its own weight, is a mere theoretical problem. In a real arch, the load is defined by the lines of intrados and extrados. There were two basic problems: first, to find the curve of extrados for a given intrados, and, second, to find the curve of intrados for a given extrados (see Figure 2b). The solution may be obtained mathematically, but Robison (1801; Figure 2c) proposed a way to obtain the curve experimentally.

The physical interpretation of the equilibrated arch is a series of smooth voussoirs with the joints always normal to the curve of intrados. The approach lead to a certain fixed form for the transmission of the thrusts: the line of intrados. The theory gave no information about the thickness of the arch and did not explain common phenomena as the cracking of arches.

In fact, the brilliant analysis of Hooke and the crucial statement of Gregory did not lead to concrete methods for the analysis or design of real arches. Mathematicians and engineers loved the problems posed by the theory, but the incapacity of the theory to afford concrete answers was made manifest in 1801, when the Select Committee, composed of the best English engineers, scientists and practitioners, was unable to ascertain the feasibility of Telford's design for a great iron bridge over the Thames in London (see Figure 8). Quoting Peacock (1855): "The answers which were given were singularly humiliating to the pride of philosophy: they were not only altogether at variance with each other, but in very instance incomplete and unsatisfactory" (pp 422-423).

France: The Joint of Rupture Theory

In France, the first contribution to arch theory is contained in La Hire's *Traité de mécanique*, published in 1695. The problem he posed was what load should sustain a semicircular

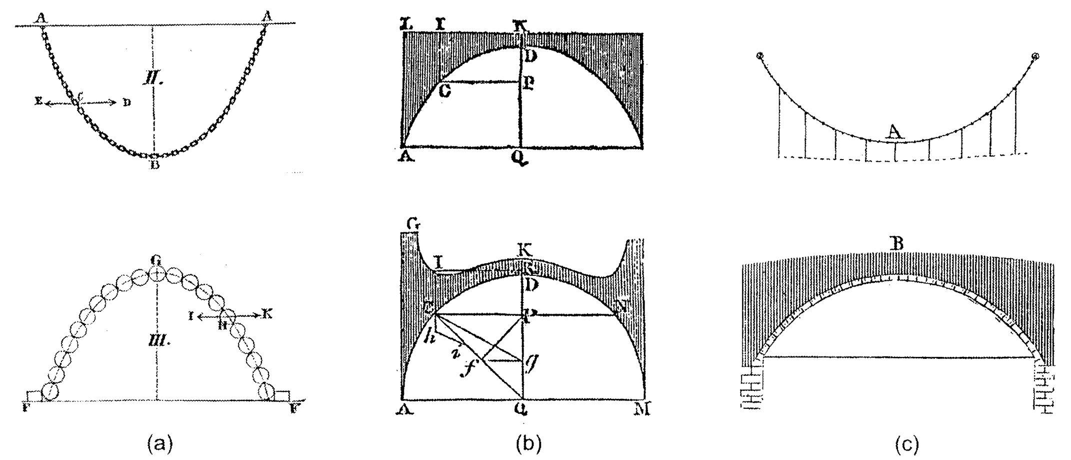


Figure 2: *The two main problems of the equilibration theory: (a) to find the curve of extrados for a given intrados, and (b) to find the form of an intrados for a given extrados (Hutton, 1812). In figure (c), the model suggested by Robison (1801) to solve the second case (drawing from Young, 1807).*

arch made of frictionless voussoirs to be in equilibrium. To solve the problem, he used the polygon of forces, and it turned out that for the arch to be stable, the load at the springings must be infinite. La Hire knew that, in practice, friction would prevent the sliding, but since then it was considered a good practice to increase the thickness of the arch from the crown to the springings.

The main contribution of La Hire (1712) was published almost two decades later. The work was not directed to the study of the form of the arch or of the load, but to obtain its thrust in order to calculate the depth of the abutments. La Hire noted that in a collapsed arch or barrel vault, the inferior part remains united to the abutment, marking the joint of rupture of the arch or barrel vault. The thrust must pass through this point and be tangent to the intrados. Once located, the point of rupture, the calculation of the thrust follows easily, establishing the equilibrium of the upper part (Figure 3a), and the depth of the buttress can be calculated. La Hire did not locate the point of rupture, and this uncertainty hindered its practical application.

Bélidor (1729) was the first to use La Hire's theory, initially in 1725 to compute a table to calculate the depth of the buttresses of gunpowder's warehouses. However, it is in his book, *La science des ingénieurs*, where Bélidor (1729) modified La Hire's approach so that it could be directly applied by any engineer. He fixed the position of the point of rupture half way between the crown and the springings, and displaced the thrust to the middle of the joint, the direction remaining parallel to the tangent to the curve of intrados at the joint. In this way, Bélidor eliminated the uncertainties in La Hire's theory and proposed a general method for the calculation of the buttresses for any barrel vault (Figure 3b) or any combination of barrel vaults, that is, for the calculation of buildings (Figure 3c). Perronet and Chezy (1810) modified the position of the joint of rupture, and located it where the curvature changes and the thrust were applied at the intrados, as seen in Figure 3d. The theory of La Hire/Bélidor was almost universally accepted on the continent during the rest of the 18th Century and was still present in some 19th Century manuals.

The arch theory of La Hire/Bélidor was interpreted as a non-friction theory of masonry: the upper part of the vault acting as a wedge pressing outwards normal to the planes of the joints of fracture. The theory gave abutments, which agreed well with the traditional proportional rules, and no safety factor was used.

In 1730, Couplet published the first memoir on arches considering friction. He realized, crucially, that when the thrust touches the intrados, a hinge forms. He studied correctly the limit thickness of arches, as illustrated in Figure 4a. In this way, he was able to calculate the limit thickness of circular arches with great approximation. However, to calculate the buttress of a stable arch, he fixed the thrust in the middle of the joint at the crown and the springings, establishing the global equilibrium of a half arch, as shown in Figure 4b.

The essays of Danyzy (1732), with small plaster models or arches demonstrated the correctness of Couplet's approach, showing convincingly the impossibility of sliding and the formation of hinges between the stones (Figure 4c). In 1800, Boistard (published in Boistard, 1810) made a fuller series of tests with arches of 2.60 m span, and these tests were considered definitive (Figure 6).

In 1773, Coulomb exposed the method to be followed (employing maxima and minima) to locate correctly the position of the joint of rupture (*mM* in Figure 5a) and calculate the thrust of the arch. He considered all possible modes of collapse at the joint: sliding upwards or downwards, or overturning upwards or downwards. However, he noticed that in practice, due to the high friction coefficients of masonry, only the failure by overturning should be considered, as Couplet did. (For a complete discussion of Coulomb's arch theory, see Heyman (1972).)

Coulomb's memoir was forgotten for almost fifty years. In 1820, the French engineer Audoy rediscovered the approach of Coulomb and wrote and solved the formulae for the most usual profiles of arches (*Figure 1*, in Figure 5b). When he applied the calculated thrusts to buttress design, he noticed that the dimensions obtained where much too slender and that some safety coefficient had to be used (see *Figure 6* in Figure 5b).

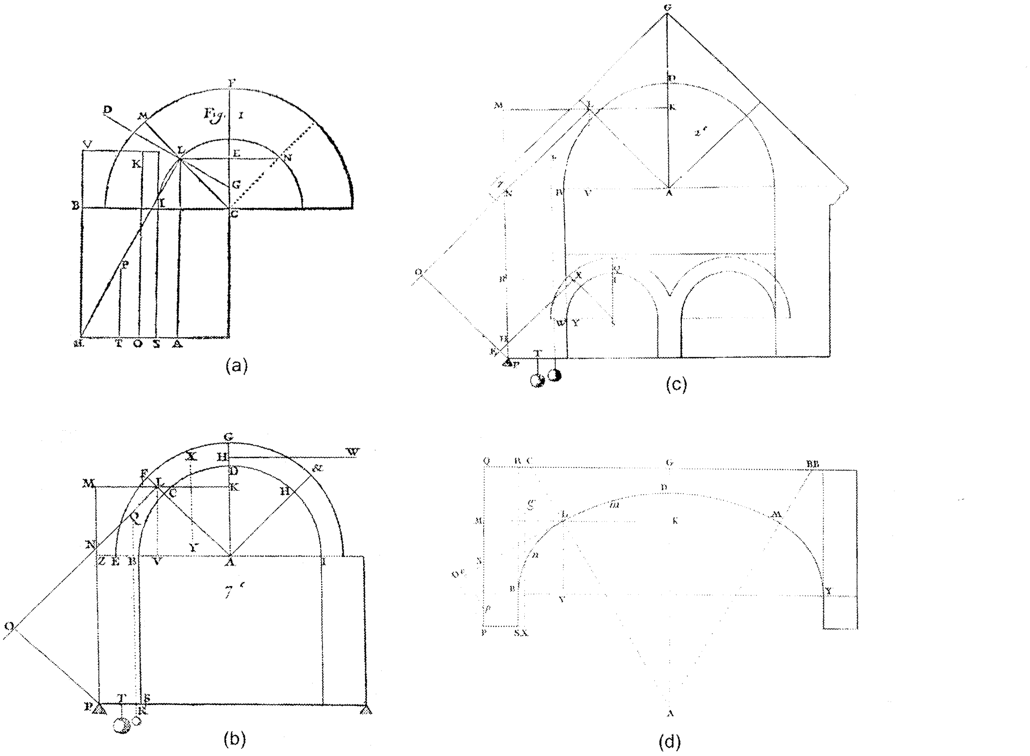


Figure 3: *La Hire theory to obtain the thrust of arches:*
(a) original theory (La Hire, 1712), (b) Bélidor's (1729) modification, (c) analysis of a vaulted building (Bélidor 1729) and (d) Perronet's adaptation to calculate the thrust of surbaissée arches (Perronet & Chezy, 1810).

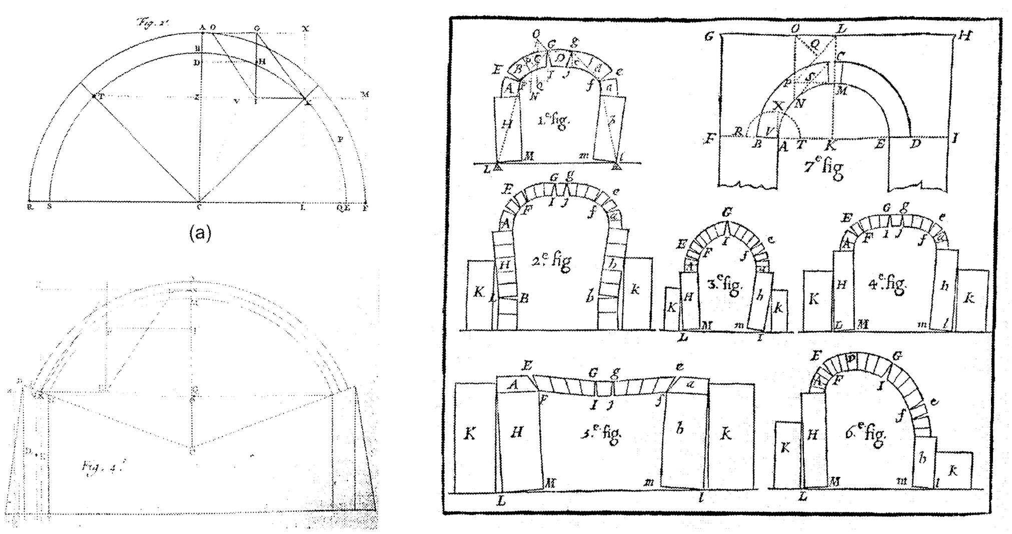
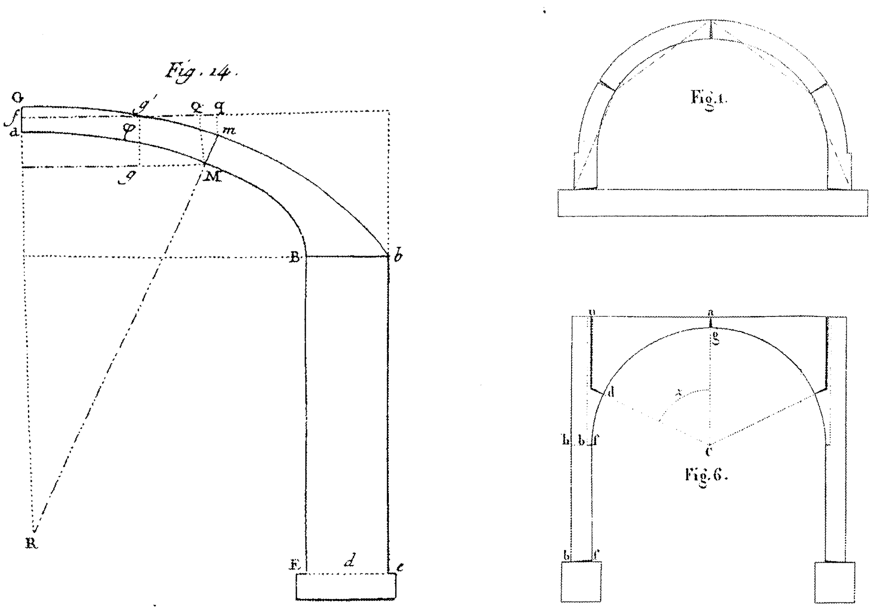


Figure 4: *Couplet's friction theory of arches:*
(a) limit thickness of a semicircular arch and (b) abutment for an arch (Couplet, 1730);
(c) tests on plaster models to demonstrate the collapse mode of arches (Danyzy, 1732).

Audoy proposed to multiply the horizontal thrust at the upper joint for a certain numerical value to obtain a safe depth.

The theoretical approach of Coulomb was adopted and completed by Navier (1826) in his *Leçons,* where he gave the integral general expressions for vaults of any form. However, Audoy's formulae were still very complicated for practical use, and rarely was there an exact solution to Navier's integrals. Some engineers calculated, then, new tables for vault thrusts and buttress design: Petit in 1835 and Michon in 1848, among others. Poncelet in 1835 developed a graphical procedure that, being somewhat cumbersome, supposed some spare time.

In summary, the French approach was concerned primarily with the determination of the joint of rupture in order to calculate the thrust against the abutments for an arch or vault of a given form. The vault-buttress system was then in a collapse situation with five hinges, one at the keystone, two at the joints



(a) (b)

Figure 5: *Joint of rupture theory:*
(a) Coulomb's analysis to find the joint of rupture (Coulomb, 1773); (b) Audoy's application of the theory of Coulomb (Audoy, 1820).

of rupture and two at the base of the buttresses (Figure 5). This thrust was then multiplied for a certain factor (usually two) to calculate the depth of the buttress. The theory did not address the problems of arch design (profile and thickness) nor the internal forces within the arch. However, the theory provided a way to calculate safely the size of the buttresses, which is in fact, the crucial element in masonry structures.

Line of Thrust Theory

As has been argued, the equilibration theory and the joint of rupture theory were completely different; however, both were used with success. A new idea was needed to obtain a complete understanding of arch behaviour: this is the concept of the line of thrust. The *line of thrust* is the locus of the point of application of the thrusts (internal forces or stress resultants) for a given family of joints (Figure 6). The thrusts need not be normal to the joints (they only should be contained within the friction cone) and the drawing of the line permits to check the main statement about the material: masonry must work in compression and hence the line of thrust must be contained within the arch.

This idea appeared in the 1830s in Germany (Gertsner, 1831), in France (Méry, 1840) and in England (Moseley, 1835). However, it was Young, in the article "Bridge" for the Supplement to the *Encyclopaedia Britannica* published in 1817 (1824), who first exposed a complete theory of arches based in the line of thrust concept (Figure 7).

Young showed in this article a deep understanding of the concept of the line of thrust. He was the first to free the curve of equilibrium from the "straitjacket" of the intrados (or the middle line). He stated that for arches of stone or cast iron, materials with good compressive strength but low tensile strength the curve of equilibrium must lay within the substance of the arch, with some geometrical safety, i.e., the curve should not approach the borders too much. Young obtained the

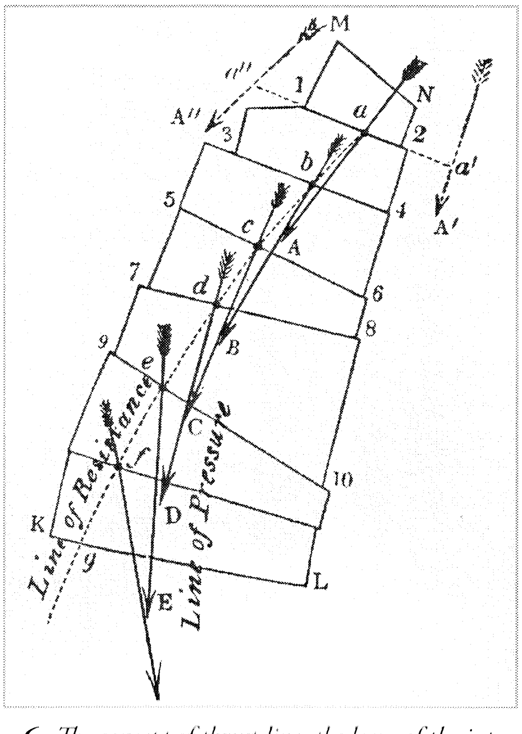
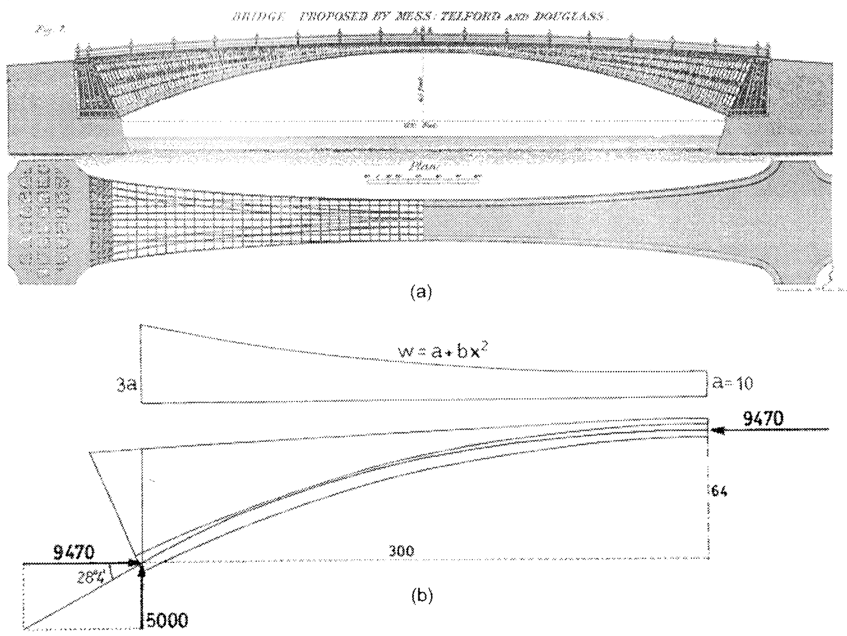


Figure 6: *The concept of thrust line: the locus of the intersection of the internal forces (stress resultants) with a given set of planes of joint that Moseley called a "line of resistance".*
The "line of pressure" is the evolute and gives the direction of the internal forces (Moseley, 1843).

general mathematical expression of the curve of equilibrium for different types of loads with a view to its application in bridge analysis. His use of a simple parabolic load is remarkable for its simplicity and applicability in most cases. In his analysis of Telford's design for an iron bridge of 600 feet (183 m) over the River Thames in London, he considered also, for the first



curve of equilibrium:

$$y = \frac{1}{1875}x^2\left(1+\frac{1}{270{,}000}x^2\right)$$

Distance x.	Versed sine of the intrados.	Versed sine of the circular arc.	Ordinate y.
50	1.73	1.71	1.34
100	6.94	6.82	5.38
150	15.66	15.43	13.00
200	28.13	27.70	24.50
250	44.42	43.81	41.01
300	65.00	64.00	64.00

(c)

***Figure 7:** Young's (1817) analysis of Telford's design for an iron arch bridge of 600 feet (183m) (Huerta, 2005b): (a) the design presented by Telford to the Select Committee (1801); (b) the equation of the thrust line and the ordinates, calculated by Young (1817); (c) drawing of the thrust line by the author using Young's computations (bottom right).*

time, the influence on the stability of the arch of a point load placed anywhere on the extrados. In so doing, he devised a completely original method of obtaining the corresponding curve of equilibrium transforming that of the dead load. Finally, he made a thermal analysis, calculating the thrust due to an increase of temperature.

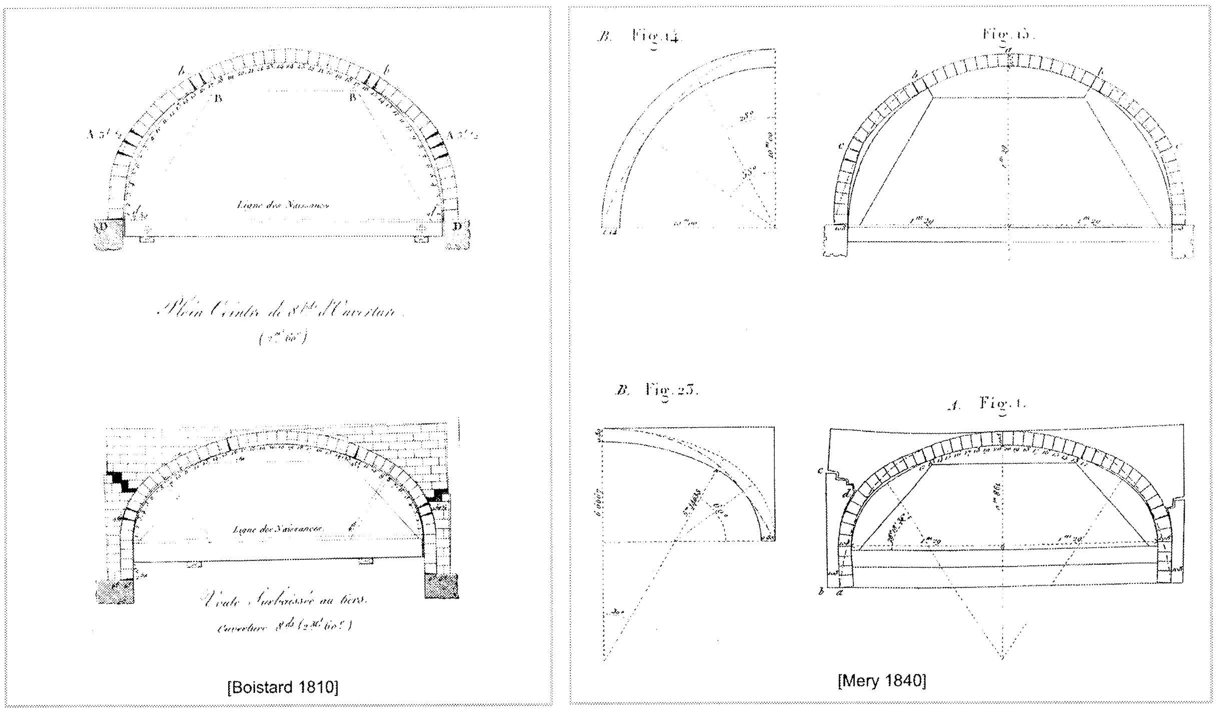
Young's analysis of Telford's design is completely correct, combining statements of equilibrium (curves of equilibrium for the given loads) with statements about the material (cast iron must work in compression; therefore the curve of equilibrium must lie within the arch). The notion of a geometrical factor of safety was implicit in many of his statements.

However, Young's contribution, written in his characteristic laconic and obscure way, was not understood at his time, exerted no influence and has been recognised only recently (Huerta, 2005b; Kurrer, 2008). On the other hand, the work of Moseley (1835) and Méry (1840) exerted an enormous influence. Both related the lines of thrust with the formation of collapse mechanisms. Méry compared with care the results of the collapse experiments made by Boistard in 1800 and interpreted them within the new theory, drawing thrust lines for the collapse state of the models and calculating the limiting thickness of the corresponding arches (see Figure 8). The concept, then, results in a fusion of the two theories. Other works followed. In England, Barlow (1846) proposed a graphical method to draw the line and devised a series of ingenious experiments to demonstrate "the existence in the practice" of such a line.

The line of thrust approach produced an enormous advance in the understanding of arch design and analysis. Besides, it made evident not only the existence of two limits for the value of the horizontal thrust (already predicted by Coulomb, 1773), but that between these two extreme situations an infinite number of lines of thrust may be drawn within the arch. Méry (1840) expressed this clearly in one of the drawings of his memoir (Figure 9). What is then the actual or true line of thrust?

Moseley (1843) was the first who tried to solve the problem applying his Principle of Least Resistance and established that the true line was that of minimum thrust. In Germany, Scheffler based his theory of vaults, published in 1857, on Moseley's approach, expanding it and giving numerous analyses and examples of applications. Scheffler's theories exerted great influence on many authors in both Europe and America. However not all authors considered it correct. For example, in 1866 Culmann tried to fix the position of the line of thrust enunciating his *Prinzip der kleinsten Beanspruchung,* which said that of all the possible lines, the actual or true was that which separated the least from the middle line of the arch.

Other solution was to design the arch so that the middle line coincided with the line of thrust. Then, it was supposed that the thrust line had to follow the path marked by the middle line. This approach supposed, in some sense, a return to the old Hookean theory of arches. Hagen in 1844 followed this approach in Germany. In France, Villarceau (1853) calculated the profiles for the arches of bridges subject to *hydrostatic loads* (the fill acting like a fluid against the extrados of the arch). In



(a) (b)

Figure 8: *Line of thrust theory:*
(a) Boistard collapse test of arches of 2.6m span (Boistard, 1800);
(b) Méry's interpretation of Boistard's test using the line of thrust theory (Méry, 1840).

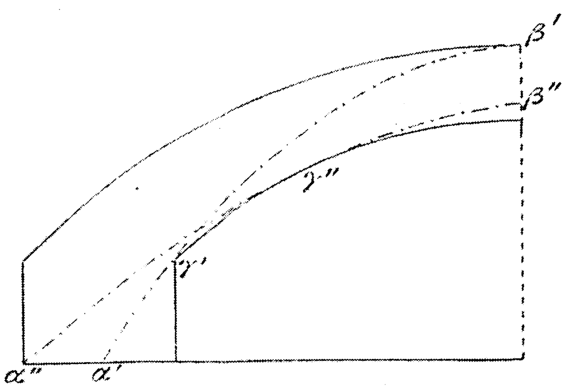


Figure 9: *Lines of maximum and minimum thrust in an arch (Méry, 1840).*

the last quarter of the 19th Century, a more drastic solution was used: to build three hinges within the masonry arch, so that the position of the line was fixed.

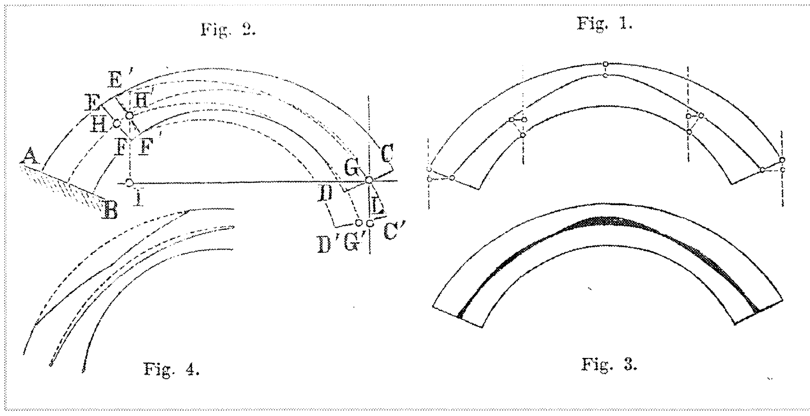
Elastic Analysis of Masonry Arches

Poncelet, already in 1852, suggested that the solution to the problem would be to consider the masonry arch as an elastic curved bar. The theory of elastic arches was then sufficiently developed. Navier in his *Leçons* of 1826 included the elastic analysis of parabolic ribs, and Bresse (1848) published the elastic analysis for a two-hinged iron arch, though his main and definitive contribution was not published until six years later (Bresse, 1854).

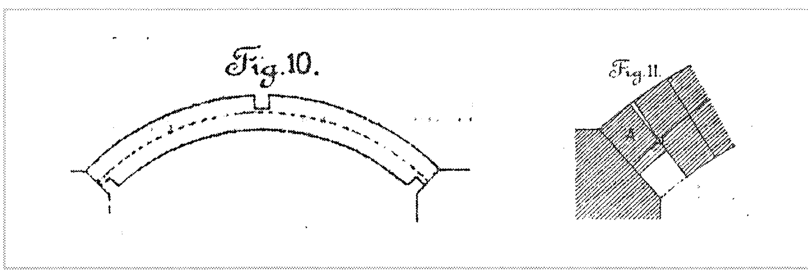
As the elastic theory was available, why not follow Poncelet's suggestion? In fact, it took almost twenty years for engineers to do so. The reason was, obviously, the difference between the actual structure of masonry arches with the ideal properties of an elastic curved bar. Usually, in engineering manuals, elastic arches (made of wood or iron) where treated separately from rigid arches made of masonry.

It appears that the Spanish engineer Saavedra made the first elastic analysis of a masonry arch in 1860. Rankine (1862), Winkler (1879-80) and others considered the problem in the 1860s, but the first complete discussion of the problem of applying the elastic analysis to masonry arches is due to Winkler (1879-80). It is remarkable that at the same time, Castigliano (1879) calculated elastically a masonry arch, the Ponte Mosca in Torino, considering the masonry with no tensile strength.

Winkler (1879-80) discussed the problem of assigning definite material properties and boundary conditions in real masonry arch construction. The masonry arch experienced perturbations (*Störungen* in German): the incomplete setting of the mortar, changes in temperature, slight yielding of the centrings during construction, and, most important, the yielding of abutments after the decentring of the arch, would lead to visible cracks and to great changes in the line of thrust. Winkler proposed to calculate the arch for a certain ideal situation: the mortar had completely set uniformly in all the joints, the centering was completely rigid, the temperature remained constant and the abutments were infinitely rigid (the arch is *encastré* in both springings). Winkler called this situation of the arch, the normal situation (*normaler Zustand*). The elastic calculation of the arch for this normal state will give as a result the correct line of thrust (*die richtige Stützlinie*). He demonstrated that the elastic line of thrust would be that which approximated more to the centre line (Figure 10a). However, Winkler gave afterwards a series of instructions to try to maintain the line in his position, that is, to avoid the effect of the perturbations (Figure 10b).



(a)



(b)

Figure 10: *Line of thrust:*
(a) determination of the true line of thrust; (b) practical measures to reduce the influence of the perturbations (Winkler, 1879-80).

Engesser (1880) discussed the effect of yielding of the abutments in the position of the line of thrust. He explained the steps to obtain the line of thrust in this situation and concluded that the line of thrust will move upwards looking for the position of minimum thrust compatible with the strength of the masonry.

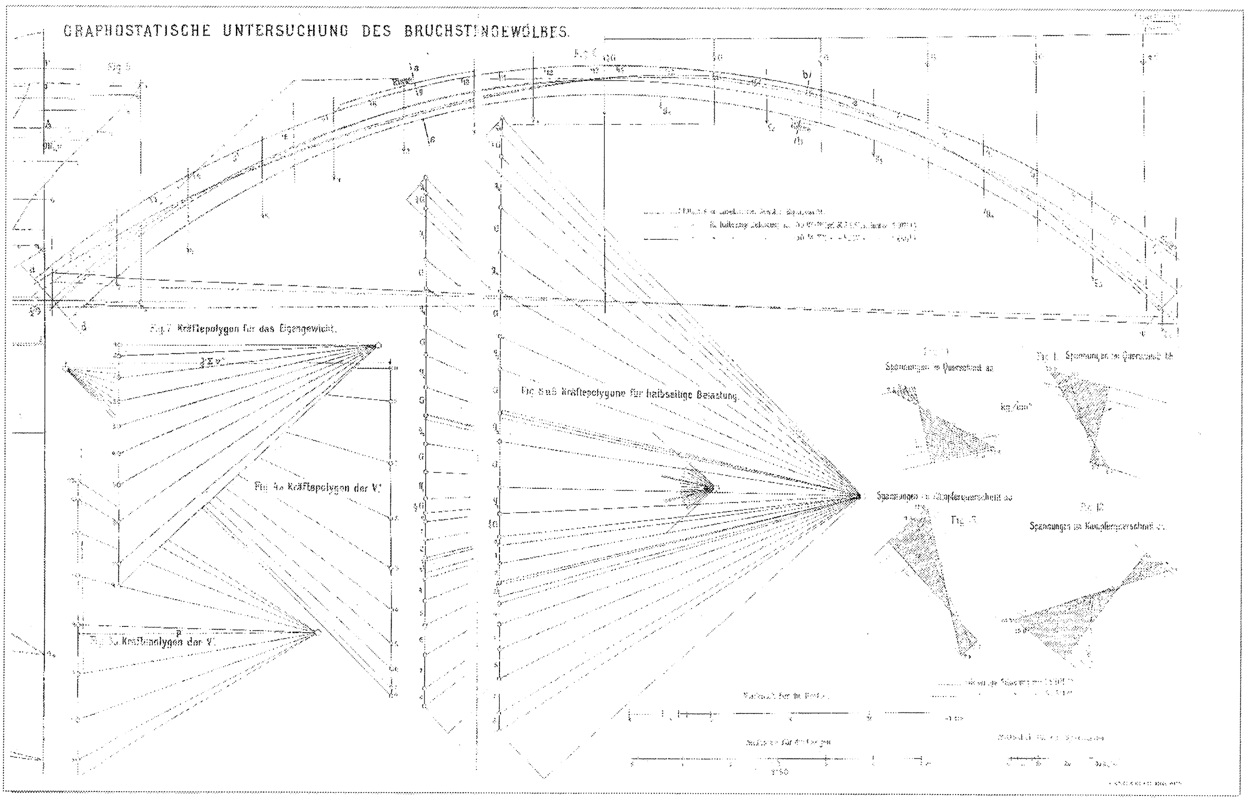
Notwithstanding the truth of Engesser's assertions and the evident influence of the *Störungen* in the normal state, engineers accepted the elastic approach as the best approach. It agreed with the new elastic theory and eliminated the uncomfortable existence of infinite solutions. As Hertwig (1941) pointed out, after Winkler, all the new contributions to arch theory tended to a simplification of the calculation process by a clever selection of the unknowns. Eventually, in 1895, the Austrian Society of Engineers and Architects made what were considered the definitive essays on arches of 23 m span. In the masonry arches, it was noted that the arch remained uncracked until a certain value of the load, being the deformations very nearly proportional to it. The increasing of the load lead, then, to greater deformations and cracking, non-proportional displacements, until the arch finally collapsed. The form of the cracks and the thrust lines drawn pointed to the formation of collapse mechanisms (Figures 11a-c), of the type described by Danyzy (1732) and Boistard (1812) a hundred years before. Besides, the curve of load-displacement marks clearly the value of the collapse load (Figure 11d). The Austrian Engineers focused their attention only in the first part of the load-deformation curve, which was very nearly straight, demonstrating the linear-elastic behaviour of the arch. There were other problems. The materials of the arches were tested in laboratories and their modulus of elasticity was registered. The elastic analysis of the arches made possible, also, to calculate the modulus of elasticity that the masonry should have for the deformations observed. The discrepancy was enormous: the arches were much more flexible than the material tests would have predicted. The Commission, then, gave some rough numbers for the values of the modulus of elasticity to be used in the calculation of masonry arches for the materials employed in the tests and recommendations addressed to avoid Winkler's *Störungen.*

Graphical Statics: The Practical Approach

The elastic analysis of arches was considered, then, the best approach as it was supposed to permit to know the position of the true line of thrust and therefore to know the actual internal forces and stresses in the arch. However, in practice the usual procedure was to check the stability by graphical statics. An arch was considered safe if it was possible to draw a thrust line within its middle third, as this was the geometrical condition that leads to a distribution of compressive stresses in all the section.

This proposition was enounced by Rankine (1858) as a theorem: The stability of an arch is secure, if a linear arch (a line of thrust), balanced under the forces which act on the real arch, can be drawn within the middle third of the depth of the arch ring. Indeed, if the thrust were contained within the middle-third of the section, there was no possibility of forming any cracks; however, the impossibility of the existence of another line of thrust leading to a collapse mechanism was not demonstrated. The same approach to arch design can be found in many engineering handbooks.

We have seen that some methods of graphical construction of lines of thrusts were developed around 1850, but they did



(a)

Figure 11: *Tests on a masonry arch of 23 m span:*
(a) Graphical calculation of line of thrust; (Austrian Society of Engineers and Architects, 1895).

not permit the study of a whole family of lines in equilibrium with a certain series of loads. A general way to handle easily the drawing process depended in the separate use of funicular and force polygons. The first to expose in a rigorous way the relationship between lines of thrust and funicular polygons, noting the differences, and remarked that graphical methods may be employed with advantage was Rankine in the 1850s, though he employed mainly analytical methods to study linear arches.

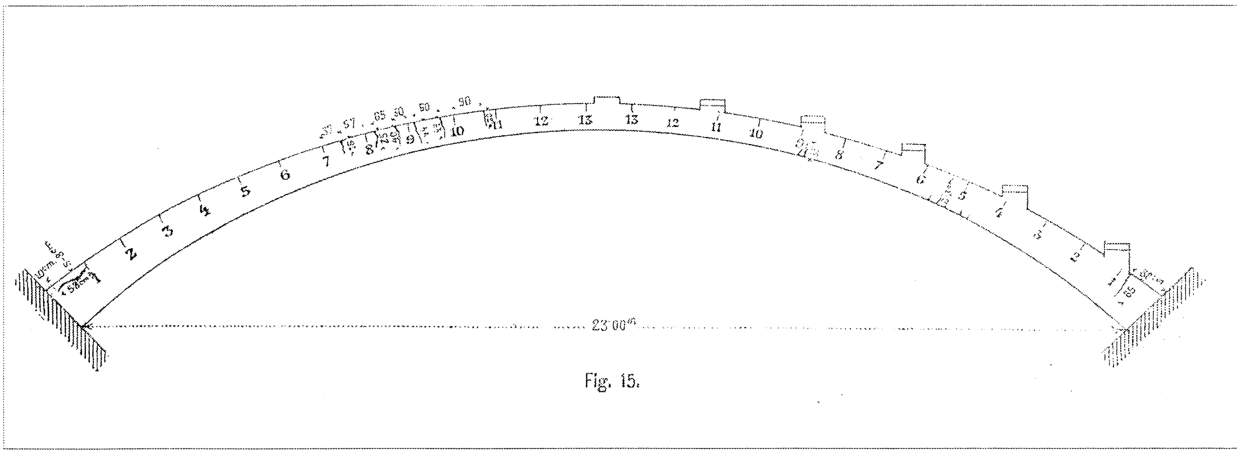
The graphical analysis of structures was proposed systematically by Culmann in 1866. He made popular the use of funicular polygons and force polygons in arch analysis (cf Maurer, 1998). This led to easy, practical methods to check the stability for any arch or buttress, and had extraordinary success. Even in the 1910s, when elastic analysis was considered the best approach, the great French engineer Séjourné (1913-16) said: "Pour une très grande voûte, il est plus sûr, plus clair, de construire ses courbes de pression, puis de tracer un intrados et un extrados qui les encadrent au mieux" (Vol. 3, p. 374).

However, around 1900 it was difficult to criticize the elastic theory, which occupied the intellectual efforts of the best theoreticians. A few authors were still critical to the elastic analysis. This is the case of Swain (1927) who in his book on graphical statics enumerated a series of false assumptions in the application of elastic theory to masonry (cf Foce, 2005). Many of them corresponded to the effect of Winkler's (1879-80) *Störungen.* (Swain was a disciple of Winkler in the 1880s and contributed to diffuse the German theory of structures in America.) In fact, Swain had the intuition of the Safe Theorem, as he stated (without proof): An arch would be stable if a line of resistance could be drawn inside the arch ring.

By 1900, the graphical analysis was the standard tool to check the stability of any arch, and elastic analysis was employed as a final check in the most important bridges. The procedure represented in Figure 12, extracted from one of the most popular German treatises (Lauenstein & Bastine, 1913) expressed clearly the advantages of this kind of approach.

Figure 12a illustrates a bridge design whose stability could be checked by these methods. The structure is divided in several parts or blocks. In Figure 12c, the stability of separate blocks, the minor arches, is studied and their thrust calculated. Then, in Figure 12b, the equilibrium of the main arch is represented. By simple inspection can be seen that the line of thrust is within the middle-third.

An exact elastic analysis of such a complex structure was impossible at the time. Nonetheless, the correct approach was to make an elastic analysis. Engineers carried complicated calculations as far as they could, in absence of computers. The advent of the computer in the 1960s and the diffusion and cheapening of finite element method (FEM) packages in the late 1980s made routine the elastic analysis of masonry bridges (it appears that the first study was by Towler in 1981). However, Winkler's "perturbations" continued to exist and produced great deviations from the calculated stress state in the



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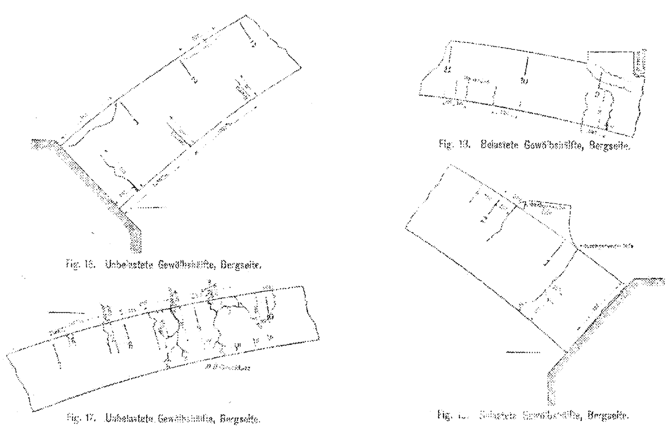

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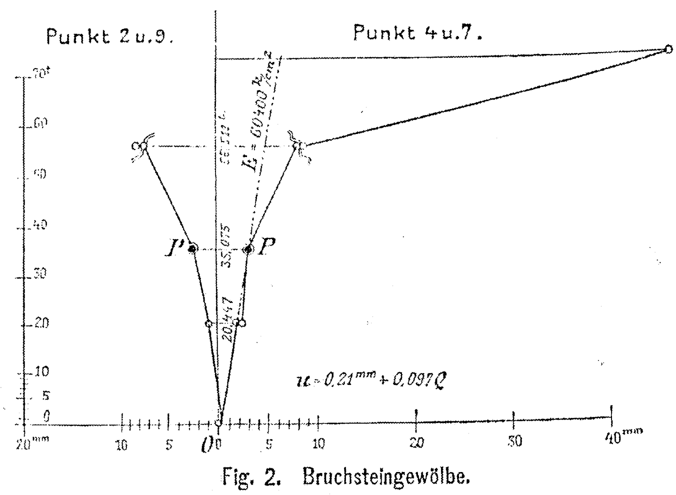


(d)

Figure 11 (continued): *Tests on a masonry arch of 23 m span: (b) and (c) Cracking of the arch; (d) Curve of load-displacement (Austrian Society of Engineers and Architects, 1895).*

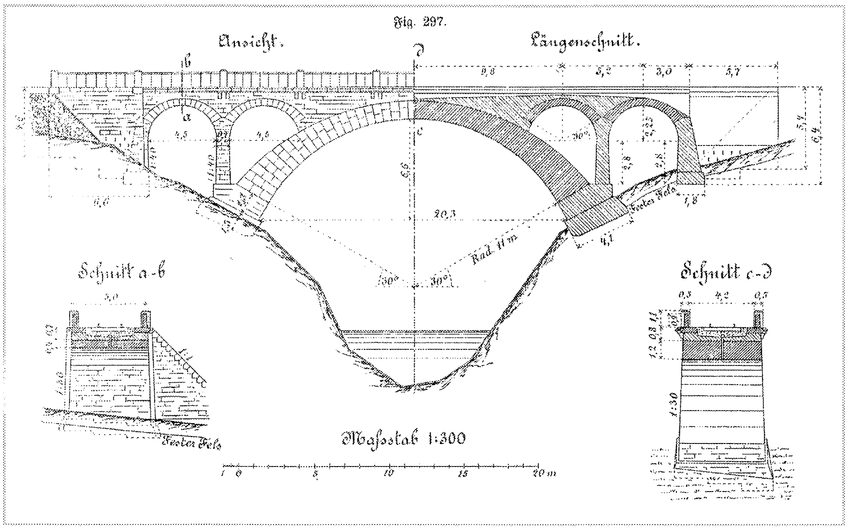
structure. As we shall see, this problem was eventually solved within the frame of Limit Analysis.

The Theory of Masonry Vaults and Buildings

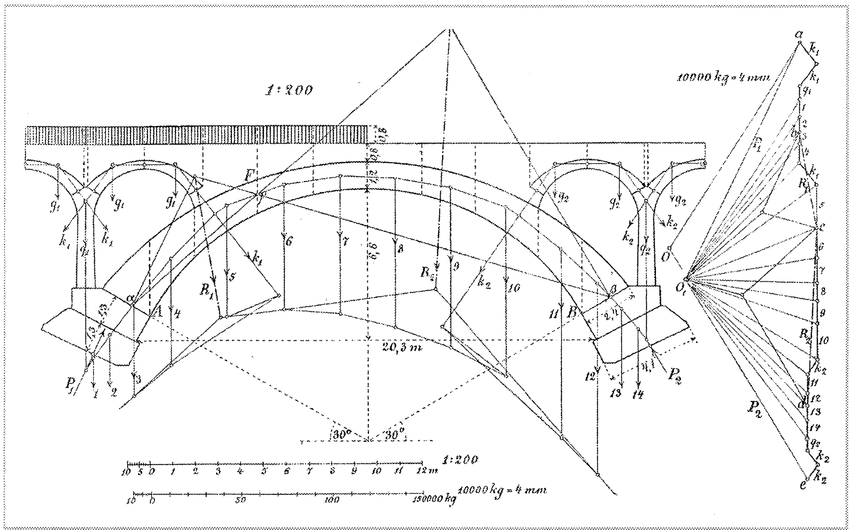
Although the history of the theory of the masonry arch is quite well known, the development of the theory of spatial masonry vaults is only roughly known (a brief outline was given in Huerta, 2004). The first scientific design for a real building was Hooke's proposal, working with Wren, for the dome of St. Paul's Cathedral in London. Heyman (1998a) has shown that Hooke thought that the cubic parabola will be the best form for a dome and, indeed, Heyman discovered that precisely this curve appears in one of the preliminary designs for the dome. The first memoir dedicated to dome design may be attributed to Bouguer in 1734. He studied the possible forms of stable domes, considering no friction, i.e., with the thrusts always tangent to the dome middle surface, but he took into account the hoop forces.

However, perhaps the work that had more influence in the development of dome theory is the treatise of Frézier of 1737-1739. He studied the composite, i.e., spatial vaults. Frézier is right when he stated that this problem was addressed in a general way for the first time. He used what is known today as the *slicing technique*. He imagined the vault composed of a series of elementary arches and this permitted him to compare the thrust of the composite vaults with those of barrel vaults of the same profile, this last thrust he considered as known. The combination of the frictionless approach of Bouguer with Frézier's slicing technique opened the door to purely mathematical research on dome design. The works of Bossut (1776), Salimbeni (1787) and Bérard (1810), all published around the end of the 18th Century, may be classified within this approach. This research was never applied in the real design of domes.

The first analyses of the stability of a real dome were made in the 1740s, as a result of the expertises ordered by Pope Benedict XIV, to explain the origin of the cracks observed in the dome of St Peter in Rome (Figure 13a). The first expertise was written by the *tre matemattici* (Le Seur, Jacquier & Boscovich, 1742; see Figure 13b). The three mathematicians studied with great care the cracks and leanings of both the dome and drum,



(a)



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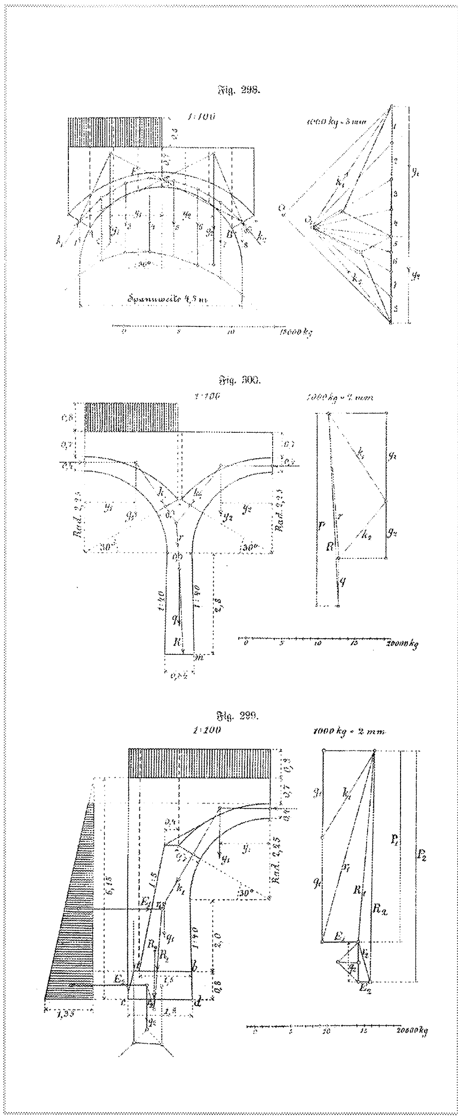
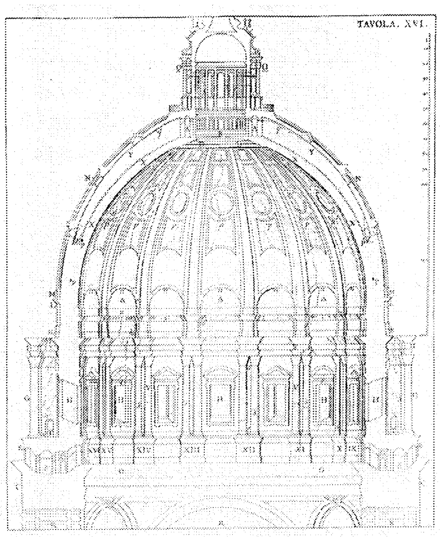


(c)

Figure 12: *Graphical analysis of a masonry bridge circa 1900 (Lauenstein & Bastine, 1913).*



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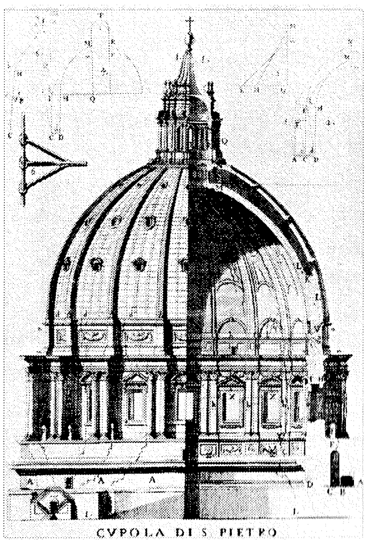

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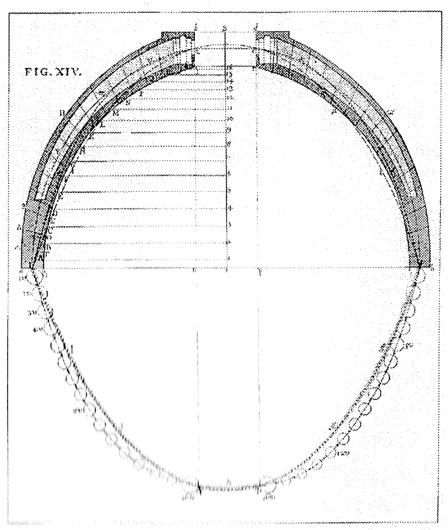


(c)

Figure 13: *The dome of St. Peters:*
(a) damages in the intrados ca 1740 (Poleni, 1748); (b) analysis of the tre matematici using the principle of virtual work in 1742 (Le Seur et al., 1742); (c) equilibrium analysis of Poleni using a hanging model in 1743 (Poleni, 1748).

and concluded that the dome cracks were a consequence of a slight yielding outwards of the drum. The mechanism of the movement is drawn in the upper right side of Figure 13b, and the three mathematicians employed the same mechanism to make and analyse the stability employing the principle of virtual work (López-Manzanares, 2001).

The second important expertise was written by Poleni in 1743 and published in his book of 1748. Poleni decided to follow the approach of Hooke and Gregory. Using the weights calculated by the three mathematicians, he imagined the dome divided in forty orange slices and each slice in sixteen voussoirs. Then he constructed a chain of beds with weights proportional to the voussoirs. He inverted the chain and verified that it was contained within the masonry of the dome (Figure 13c). Poleni concluded that the dome was safe (though he recommended adding five iron ties, which were installed by Vanvitelli (Poleni, 1748).

Boscovich, one of the three mathematicians, wrote another two expertises (cited in López-Manzanares, 2005): the first to ascertain the safety of the great oval dome of the Cesarean Library in Vienna, and the second to check the safety of the new spire to be placed on top the Tiburio of Milan Cathedral (López-Manzanares, 2005). At the end of the 18th Century, the French engineer Gauthey wrote a dissertation on dome analysis in 1798 in the context of the debate about the feasibility of Soufflot's design for the new dome of St Genevieve in Paris where he applied correctly the slicing technique and considered a collapse mechanism by hinge formation.

Lamé and Clapeyron in 1823 (cited in Heyman, 1967a) studied the stability of a hemi-spherical dome of 1/32 of span thickness, in the context of the preliminary designs for the Cathedral of St Isaac in Saint Petersburg. They imagined that the dome would be divided in orange slices and calculated correctly the position of the joint of rupture and the corresponding thrust. However, they supposed that the upper joint of rupture would be at the keystone, and, in fact, it will be displaced some distance from it, the upper part of the dome remaining uncracked (Figure 13b-c; Heyman, 1967a).

The same year, Dietlein (1823) published a memoir on groined vaults. He followed the old theory of Eytelwein (1808) and was concerned mainly with the calculations of centres of gravity, dividing the vault in elementary slices (Figure 12a). Navier in his *Leçons* published in 1826 also treated, in a general way, the problem of the analysis of groined and cloister vaults. He studied the case of domes and dedicated attention to the problem of the iron rings and the way they should be calculated. Persy (1834), Kobell (1855) and Scheffler (1857; see Figure 14b), also studied the way the mathematical analysis of domes, groined and cloister vaults could be made. All the cited contributions attacked the problem in a mathematical way but the proposed methods were never used in practice by architects and engineers. In contrast with this general trend, Michon (1857), made a very interesting proposal for a fast calculation of the thrust considering global equilibrium and the actual cracks at the vault (see Figure 14c).

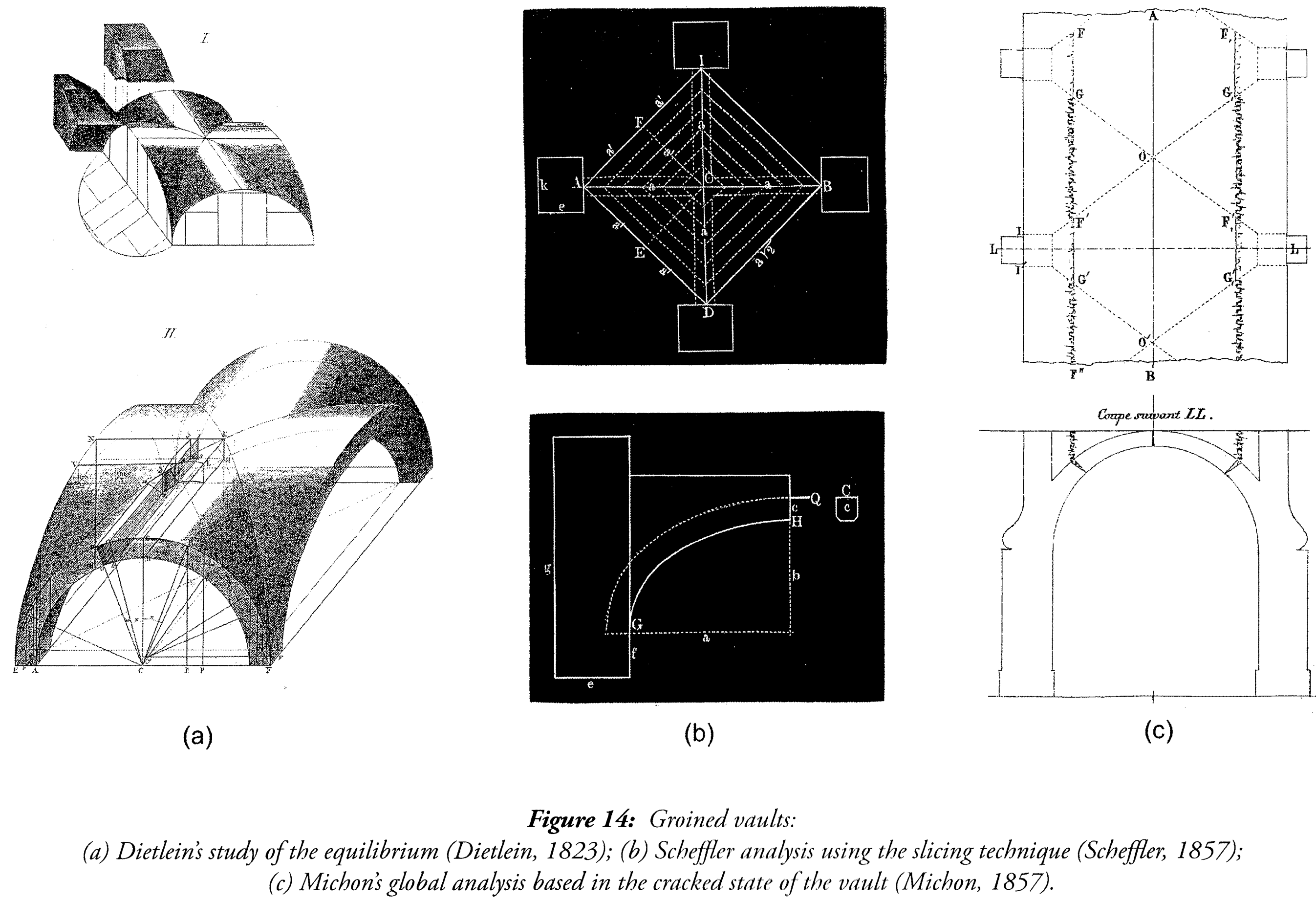
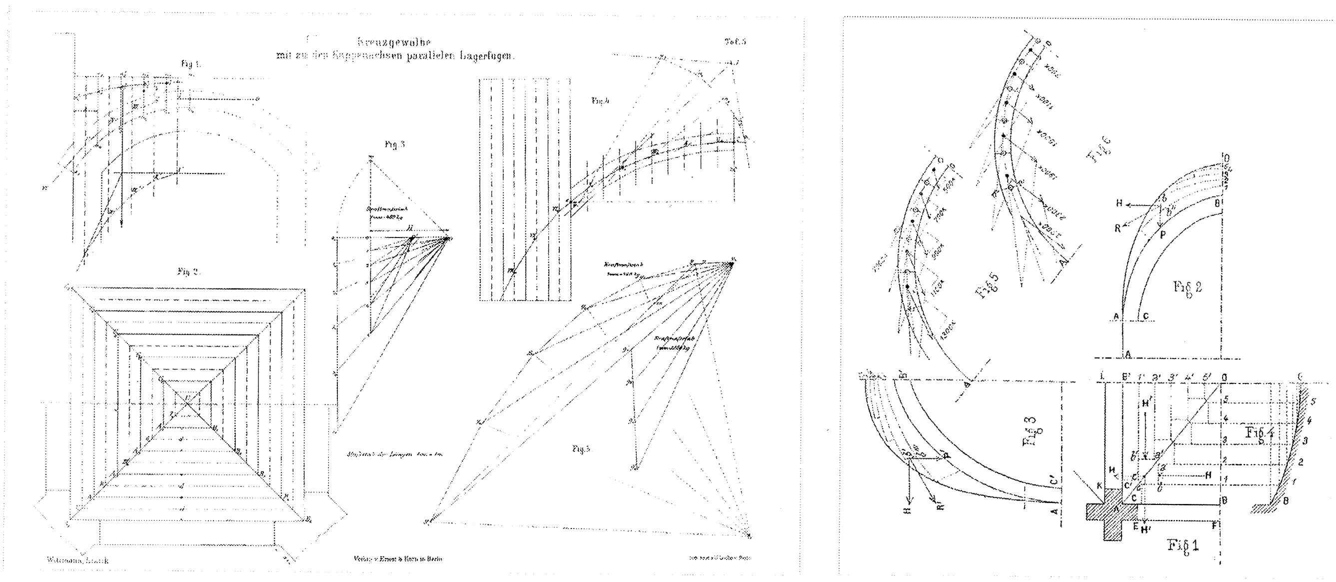


Figure 14: *Groined vaults:*
(a) Dietlein's study of the equilibrium (Dietlein, 1823); (b) Scheffler analysis using the slicing technique (Scheffler, 1857); (c) Michon's global analysis based in the cracked state of the vault (Michon, 1857).



(a) (b)

Figure 15: *Graphical analyses:*
(a) first correct graphical equilibrium analysis of a groin vault (Wittmann, 1879);
(b) graphical analysis with force resolution in the drawing by Planat (1887).

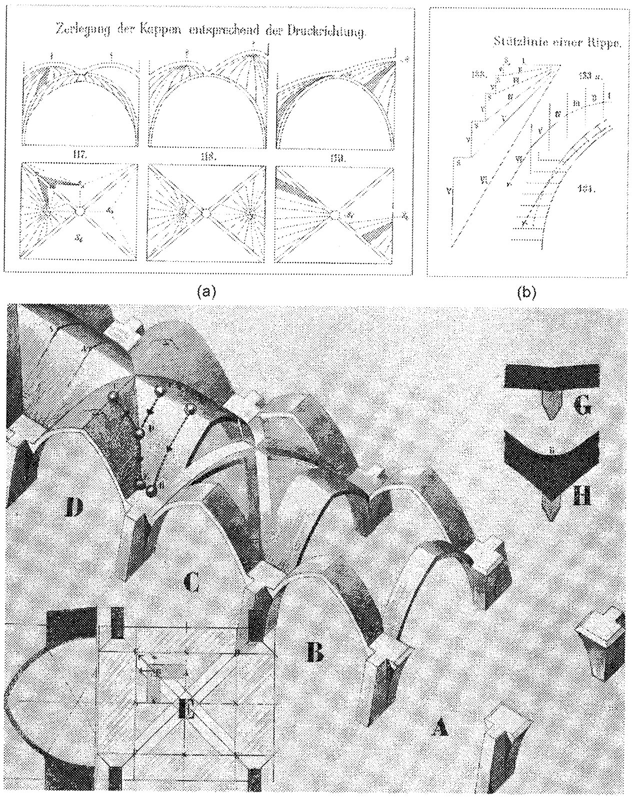


Figure 16: *Mohrmann's explanation of the slicing technique:*
(a) possible divisions of the webs;
(b) drawing of the line of thrust in the cross rib (Ungewitter, 1890);
(c) explanation of the "ball" experiment to explain the internal thrust trajectories (Abraham, 1934).

Graphical Statics of Vaults and Domes

In the 1870s, the graphical methods of arch analysis, combined with the slicing technique, permitted the analysis of any kind of vaults and the calculation of their thrust. In 1879, Wittmann published what is apparently the first graphical analysis of spatial vaults (groined vaults and domes; Figure 15a). Planat (1887) addressed the matter of vault and building analysis. Planat did not use the polygon of forces and preferred to resolve the forces on the same drawing (as seen in Figure 15b). (The different methods of cross-vault analysis have been discussed in depth by Barthel (1993).)

However, it was Mohrmann in the third edition of the *Lehrbuch der gostischen Konstruktionen* of Ungewitter (1890) who gave the most complete exposition of the static study of masonry vaults, addressed to the design of cross vaults for neo-Gothic churches. The first two chapters on vaults and buttresses, completely newly written by Mohrmann, contain the best exposition of the building and structural analysis of masonry Gothic architecture.

To obtain the thrust of a Gothic vault, Mohrmann (in Ungewitter, 1890) first discussed the use of the slicing technique. The webs are divided in elementary arches or strips and the form of the cutting depends on the form of the vault (Figure 16a). Then the reactions of these arches are the loads on the cross ribs of the vault (Figure 16b) which are transmitted eventually to the buttresses and foundations. The form of the strips remained undefined. Mohrmann imagined that the thrust would follow the same path as a ball rolling on the extrados. The idea was revived by Abraham (1934) who made excellent drawings to explain it (Figure 16c).

The method, though clear, can imply a long series of calculations and graphical analyses, particularly when the division leads to elementary arches of different forms. Mohrmann (in Ungewitter, 1890), thus, proposed another

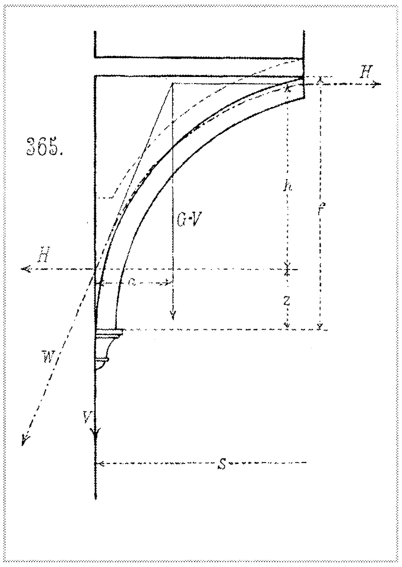


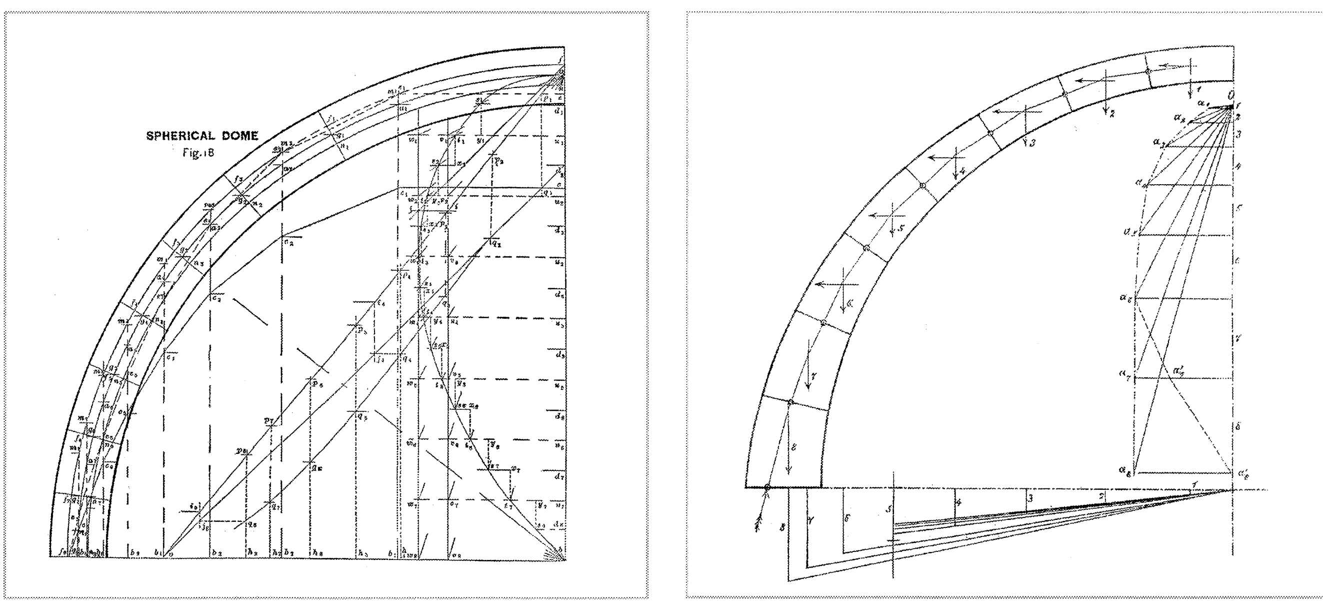
Tabelle 1.

Die Gewichte und Horizontalschübe der Gewölbe (s. Fig. 365).

Bezeichnung des Gewölbes	Gewicht von je 1 qm Grundrissfläche G_0	Hebelarm des resultierenden Gewichtes a	Hebelarm der Horizontalschübe h	Höhe des Widerlagsdruckes über Gewölbebeginn Z	Horizontalschub für je 1 qm Grundriss des lastenden Gewölbstückes H_0	Beispiel I. Gewölbe von 4 . 4 m Gewicht einer Hälfte V	Beispiel I. Schub einer Hälfte H	Beispiel II. Gewölbe von 8 . 8 m Gewicht einer Hälfte V	Beispiel II. Schub einer Hälfte H
I. Pfeilverhältnis 1:8.									
a. Kappen 1/2 Stein aus porösen Ziegeln	200				360—400	1600	3200	6400	11500
b. 1/2 Stein feste Ziegel oder 3/4 Stein porös	270	0,22—0,23 s*	0,90 f**		500—550	2160	4400	8600	16000
c. 3/4 Stein feste Ziegel oder 1 Stein porös	370		oder	1/8 bis 1/6 f	700—750	2960	6000	11800	22400
d. 1 Stein feste Ziegel od. 20 cm dick Sandstein	500	rd 2/9 s			950—1000	4000	8000	16000	30400
e. 30 cm dick Bruchstein	850		1/10 s		1600—1700	6800	13600	27200	51000
f. Ueberfülltes Ziegelgew., mit Fussb. 32 cm im Scheitel	—	0,20 s = 1/5 s			—	5800	11000	26000	46000
II. Pfeilverhältnis 1:3.									
a. Kappen 1/2 Stein aus porösen Ziegeln	230				160—180	1840	1440	7400	5100
b. 1/2 Stein feste Ziegel oder 3/4 Stein porös	310	0,19—0,21 s	0,85—0,75 f		220—240	2460	1920	9900	7000
c. 3/4 Stein feste Ziegel oder 1 Stein porös	420		oder	1/6 bis 1/4 f	300—330	3360	2640	13400	9600
d. 1 Stein feste Ziegel oder 20 cm dick Sandstein	570	rd 1/5 s			420—450	4560	3600	18200	13400
e. 30 cm dick Bruchstein	1000		3/10—1/4 s		710—750	8000	6000	32000	22700
f. Ueberfülltes Ziegelgew., mit Fussb. 32 cm im Scheitel	—	0,17 s = 1/6 s			—	7300	5200	37500	23000
III. Pfeilverhältnis 1:2.									
a. Kappen 1/2 Stein aus porösen Ziegeln	260				110—120	2080	960	8300	3500
b. 1/2 Stein feste Ziegel oder 3/4 Stein porös	350	0,17—0,20 s	0,80—0,70 f		140—160	2600	1280	11200	4500
c. 3/4 Stein feste Ziegel oder 1 Stein porös	480		oder	1/5 bis 1/3 f	190—220	3640	1760	15400	6100
d. 1 Stein feste Ziegel oder 20 cm dick Sandstein	700	= 1/6—1/5 s			280—320	5600	2560	22400	9000
e. 30 cm dick Bruchstein	1200		2/5—1/3 s		480—550	9600	4400	38500	15300
f. Ueberfülltes Ziegelgew., mit Fussb. 32 cm im Scheitel	—	0,16 s			—	8000	3800	41600	17600
IV. Pfeilverhältnis 2:3.									
a. Kappen 1/2 Stein aus porösen Ziegeln	290				90—100	2320	800	9300	2900
b. 1/2 Stein feste Ziegel oder 3/4 Stein porös	380	0,17—0,20 s	0,80—0,72 f		110—130	3040	1040	12200	3500
c. 3/4 Stein feste Ziegel oder 1 Stein porös	530		oder	1/5 bis 1/4 f	160—180	4240	1440	17000	5100
d. 1 Stein feste Ziegel oder 20 cm dick Sandstein	750	= 1/6—1/5 s			220—250	6000	2000	24000	7000
e. 30 cm dick Bruchstein	1300		rd 1/2 s		400—430	10400	3140	41500	12800
f. Ueberfülltes Ziegelgew., mit Fussb. 32 cm im Scheitel	—	0,16 s			—	10500	3500	57900	17400
V. Pfeilverhältnis 5:6 bis 1.									
a. Kappen 1/2 Stein aus porösen Ziegeln	340				80—90	2720	720	10900	2600
b. 1/2 Stein feste Ziegel oder 3/4 Stein porös	450		0,80—0,75 f		100—110	3600	880	14400	3200
c. 3/4 Stein feste Ziegel oder 1 Stein porös	650	0,16—0,19 s	oder	1/5 bis 1/4 f	150—160	5200	1280	20800	4800
d. 1 Stein feste Ziegel oder 20 cm dick Sandstein	900				210—230	7200	1840	28800	6700
e. 30 cm dick Bruchstein	1500		rd 2/5 s		350—370	12000	2960	48000	11200
f. Ueberfülltes Ziegelgew., mit Fussb. 32 cm im Scheitel	—	0,15 s			—	13000	3000	77800	17500

* s = Spannweite, ** f = Pfeilhöhe.

Figure 17: Mohrmann's table for the rapid calculation of a vault thrust (from Ungewitter, 1890).

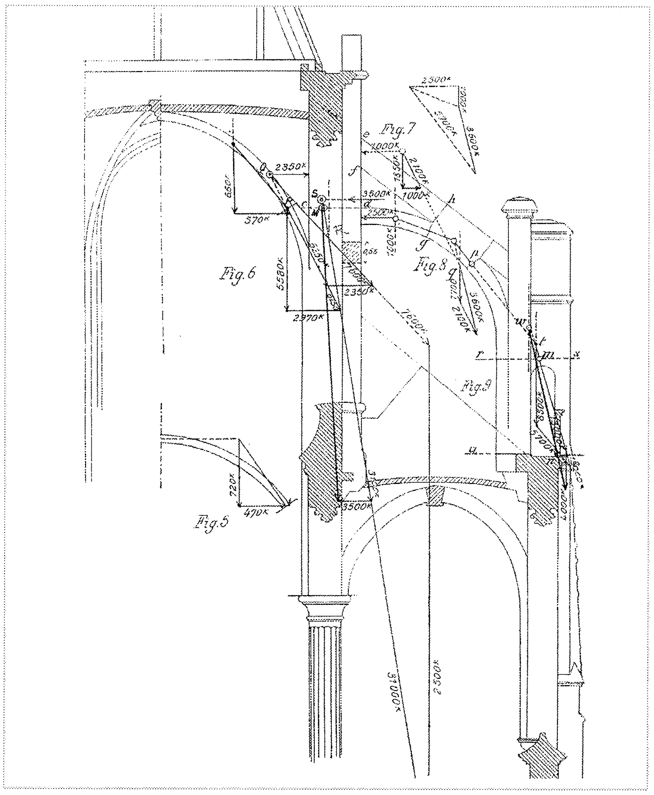


(a) (b)

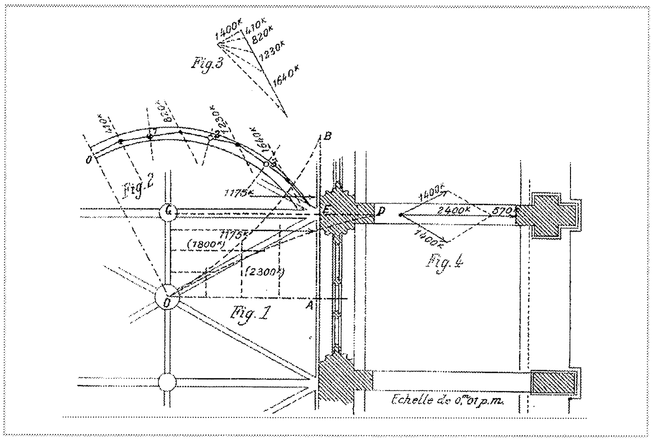
***Figure 18:** Analysis of equilibrium:*
(a) graphical method by Eddy to obtain the meridian and hoop forces in a dome (Eddy, 1878);
(b) analysis of a masonry dome considering a membrane compressive state in the upper part by Föppl (1881).

faster approach. He established the global equilibrium of a half vault for a single bay (Figure 17a) and calculated a table to obtain the thrust of vaults of different materials, loads and proportions, as seen in Figure 17b. The table can be used with advantage today and Heyman included a simplified version in his book of 1995.

The analysis of the equilibrium of domes was much easier due to axial symmetry (see Figure 18). As we have seen, the dome is imagined divided in orange slices by meridian planes and every two opposed slices form an arch. Sometimes the existence of hoop compressive forces in the upper part was considered. Wittmann (1879) devised a graphical method to construct line of thrust in this way, which was followed by other German authors (e.g., Körner, 1901 and others; cited in Wittmann, 1879). Eddy, in 1878, devised a graphical method to calculate the hoop forces (both compressive and tensile) for a membrane state (thrust within the middle surface) and Föppl (1881) applied the method to masonry domes. The method permitted the calculation of



(a)



(b)

Figure 19: *Graphical equilibrium analysis of a three-nave church by Planat (1906): (a) static study of the upper part including the vault and flying buttress; (b) calculation of the thrust of the main vault.*

domes of revolution of any form. Dunn diffused the approach in America ca. 1900 and Guastavino used it extensively for the design of thin brick domes in the first third of the 20th Century (both cited in Huerta, 2003). However, the usual approach was zero hoop stresses and the same technique as in arch analysis could then be used.

Graphical Analysis of Building

The first analysis of buildings was made in the 18th Century in expertises written to ascertain the stability of some domes (St Peter in Rome, the dome of the Cesarean Library in Vienna and the French Pantheon). All of them showed a deep understanding of the mechanics of masonry and involved the use of theories and analytical tools of great usefulness and ingenuity. However, they exerted almost no influence.

In the 19th Century, it appears that the first analysis of a whole section of a building (a Gothic church) was made by Planat in 1877, more than a century after the above cited studies. In this contribution, Planat calculated the thrust of the elementary arches of the cross vaults fixing arbitrarily the position of the thrust, following the theory of La Hire/Bélidor. Ten years after, in 1887, after making a correct graphical analysis of the vaults (Figure 15b), he attacked the problem of the analysis of whole buildings. The building was imagined divided in individual elements or blocks: vaults, flying buttresses, walls and counter-forts, etc. The stability and the reactions of these blocks were studied and, eventually, the building was mounted and a check was made that the lines of thrust were all contained within the masonry. Planat (1906) expanded his first contribution in successive editions and new books, which contained detailed analysis of complete masonry buildings (Figure 19).

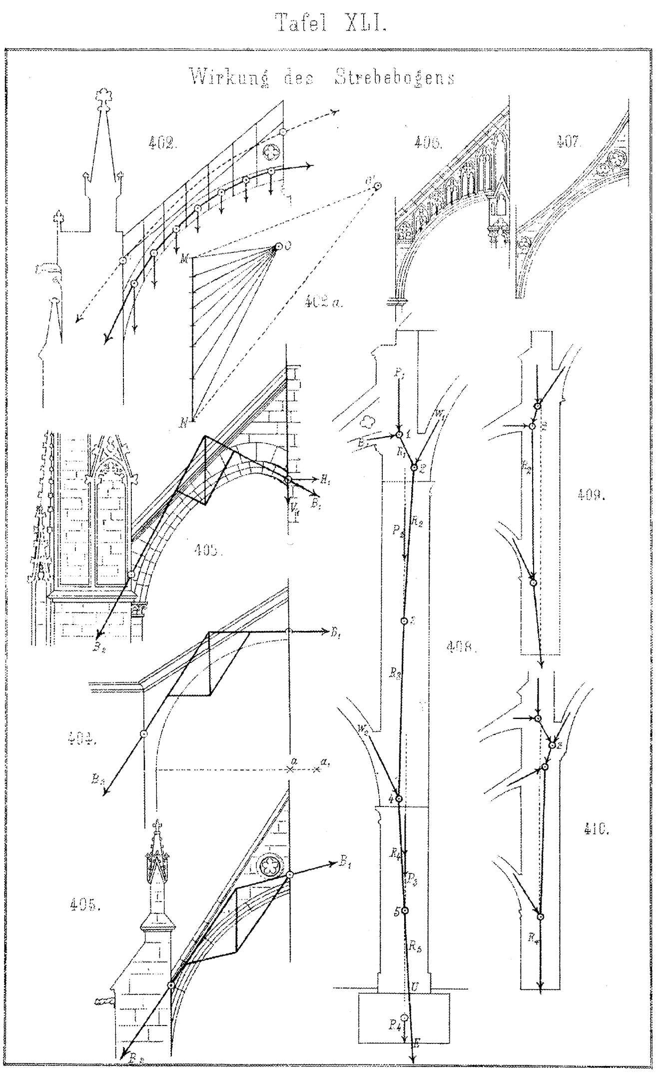
The additions of Mohrmann for the third edition of Ungewitter (1890) were more precise, entered in many details of design and, in fact, addressed in an exhaustive way the problem of the design of a whole building. Like Planat, Mohrmann made no comments about previous static analyses of Gothic structures, so it is difficult to track the development of a method, which, in the case of Mohrmann manifested itself in a state of great perfection. There is insufficient space here to explain the depth of understanding and the level of detail of Mohrmann's analysis (see Figure 18). Even today, it is still the best manual to study and understand Gothic structures. It is evident that he worked for years preparing all the material for the new edition of Ungewitter. We arrive, then, to the same date of circa 1880, when the awareness of the potential of the new graphical analysis of equilibrium was discovered by some practicing architects and engineers. More research is needed to know the details of the development.

The graphical equilibrium analysis of buildings diffused at great speed through all Europe and by ca 1900 was a common approach. In France, for example, Benouville in 1891-1892 made a complete analysis of the equilibrium of one typical bay of Beauvais Cathedral. The work of Körner (1901) deserves also to be mentioned as it contains a static study of almost every form of vault.

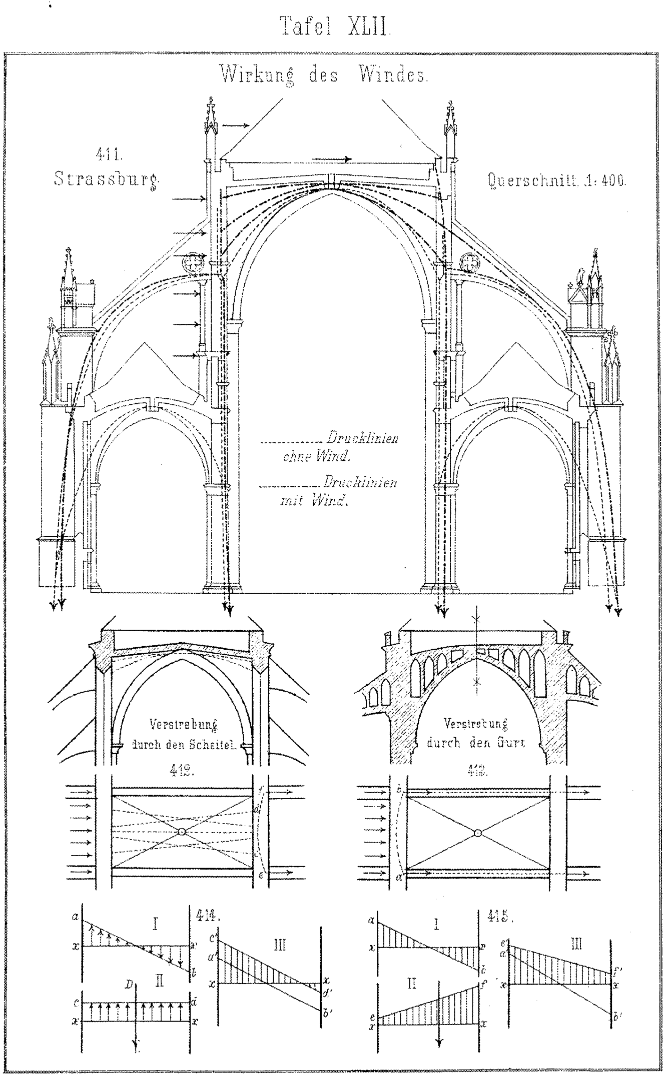
In America, also at the beginning of the 20th Century, graphical equilibrium analysis was used to check the feasibility of the modern design of old historical domed and vaulted structures. Examples were the level of detail reached in the analysis of the Chapel of St Paul at Columbia University (1904–1907; cf Goodyear, 1906) and vaults built by the Spanish architect Guastavino (Figure 21).

In Spain, Gaudí designed his masonry buildings with equilibrium methods (Huerta, 2006). Sometimes he used hanging models as in the roofs of Casa Milá, and the church of the Colonia Güell, and at other times, he used graphical statics such as for the buttresses in Parque Güell or the Church of the Sagrada Familia (Figure 22; for a history if the use of hanging models in vaults analysis, see Graefe, 1986).

One of Gaudí's disciples, Rubiò y Bellver, published in 1912



(a)



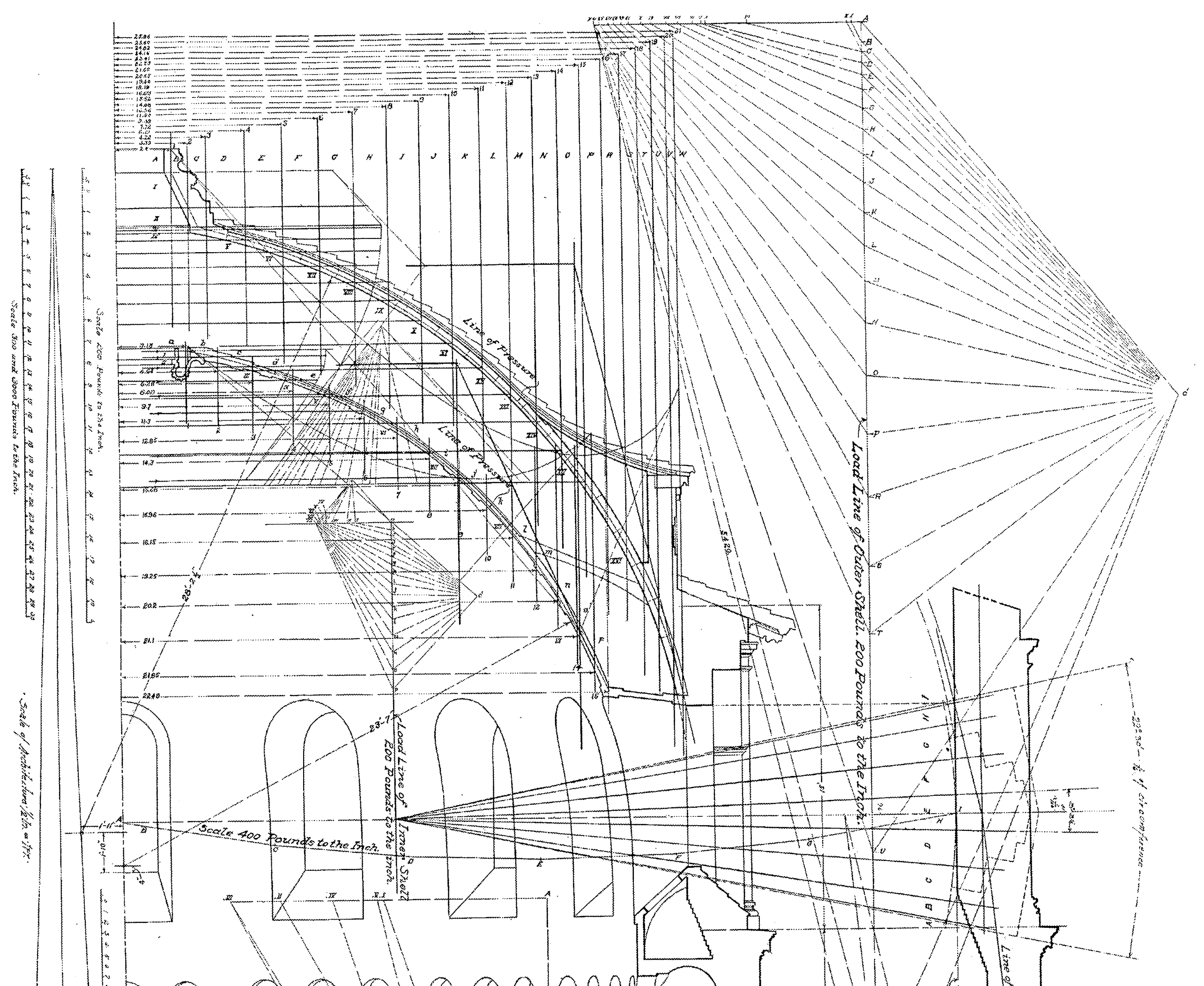
(b)

Figure 20: *Analysis Gothic churches and cathedrals by Mohrmann (in Ungewitter, 1890): (a) analysis of flying buttresses and stability of piers; (b) analysis of Strassbourg Cathedral.*

a memoir on the statics of the Cathedral of Palma de Mallorca. The study was made with great care and detailed tables gave the dimensions and loads of all the parts of the building. He then calculated the thrust of the vaults and flying buttresses following the method explained above. Now the problem was to find an equilibrium solution within the masonry. The task of Rubiò was, then, to look for a possible state of equilibrium with an almost vertical resultant going through the mass of the pillars of the nave. The structure is highly hyperstatic. The thrust of the vaults oscillate between narrow margins, but there are, in fact, three flying buttresses, two apparent and the third one being the transverse arch of the lateral nave which supports an ashlar wall, functioning as an horizontal flying buttress (Figures 21a and b). Rubiò (1912) complained about the amount of work that he needed to arrive to a satisfactory equilibrium solution. In the end, this is possible due to the great weight added on top of the nave vaults, charged with heavy pyramids (Figure 23c) and walls (Figure 23d) of ashlar masonry. The great weight on the upper vaults produced a great horizontal thrust and enormous flying buttresses transmit the thrust to maybe the biggest buttress system of the Gothic period.

Graphical equilibrium analysis continued to be the standard approach to check the stability of proposed new designs and to extract conclusions regarding the structural safety of existing buildings. For example, Rüth in the 1920s, when working in the consolidation of the Cathedral of Mainz (Rüth 1928), made this kind of analysis to calculate the vault thrusts, the equilibrium of the different parts under study and, eventually, the stresses on the foundations. In 1933, Zorn wrote a dissertation with a complete study of the equilibrium of the hall church of St. Martin in Landshut.

Because of the Second World War in Germany, many churches and cathedrals suffered great damage, many times with the complete destruction of parts of the buildings. The study of the safety of those structures was analyzed following the traditional graphical methods of equilibrium. Pieper (1950, 1983) was a central figure at this time and was involved for almost fifty years in the field. His first analysis of completed buildings (e.g., Marienkirche in Lübeck) was published in 1950 (Pieper, 1950). He obtained the thrust of the vaults by global equilibrium, calculating the form of the load on a transverse plane and looking for the minimum thrust, making the line of



STRESS DIAGRAM OF THE DOMICAL TOWER SHOWING CONSTRUCTION AND CALCULATIONS IN CONNECTION WITH THE INNER DOME, THE OUTER DOME AND THE PENDENTIVES.

Figure 21: *Graphical equilibrium analysis of the Chapel of St Paul at Columbia University (Goodyear, 1906).*

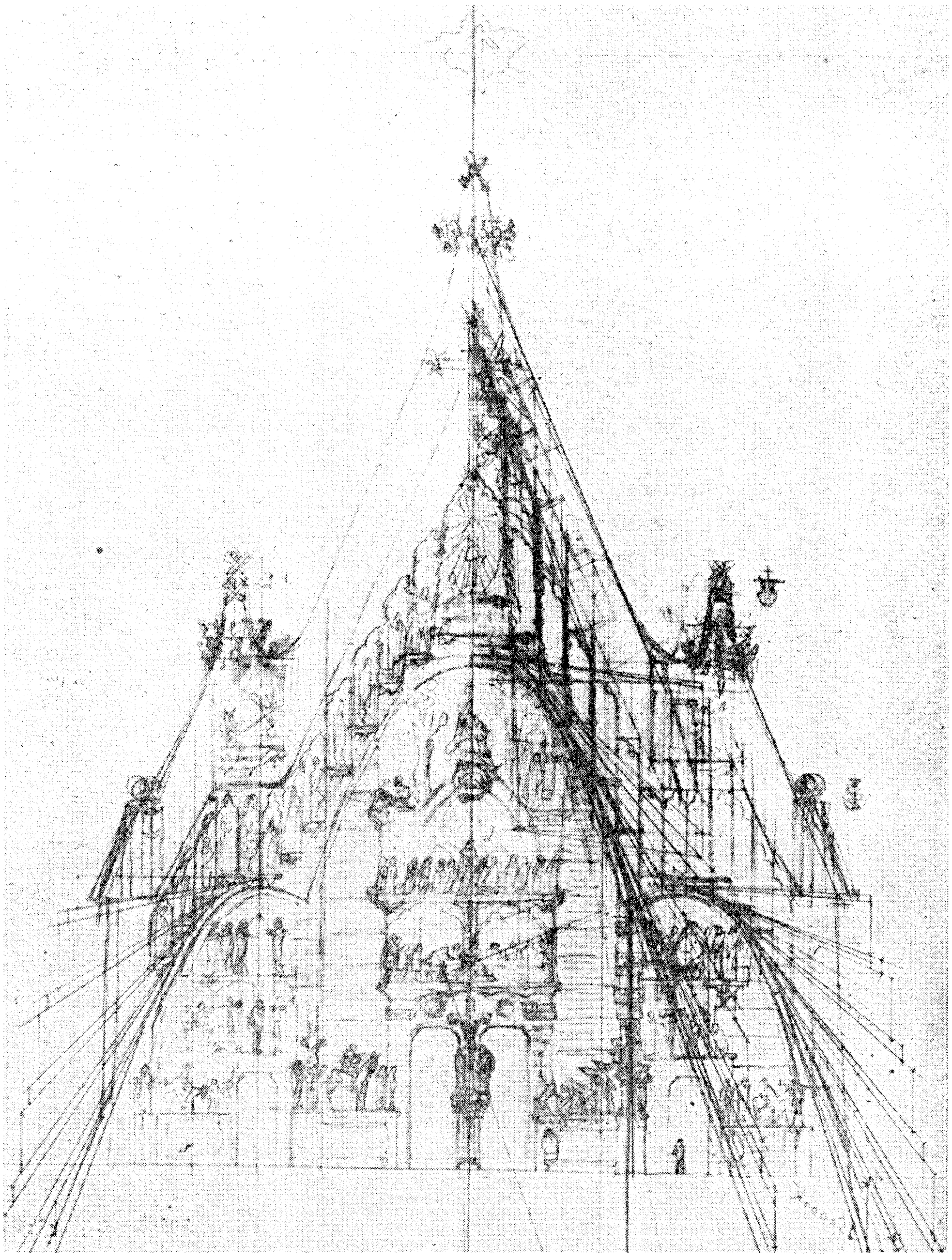


Figure 22: *West facade of the Sagrada Familia; static analysis by Gaudí (Rafols, 1929).*

thrust tangent to transverse rib. Then he obtained the minimum thrust for the flying buttresses and established the equilibrium of the whole building (Figure 24). Much of his work is compiled in a book (Pieper, 1983). The approach is always of equilibrium, though when cracking was evident, he tried to fix some hinges that make the structure isostatic. However, it is essentially the same method of Planat, Mohrmann and Rubió. Grassnick's (1963) detailed analysis for the reconstruction of the vaults of the Cathedral of Xanten was made following the same graphical analysis of equilibrium.

Classical Elastic, FEM and DEM Analyses

The equilibrium approach was considered rough, when not incorrect, by structural theoreticians between 1900 and the 1960s. An elastic analysis should have been made, but it was not applied to spatial masonry vaults (domes, cross vaults, etc.) because of the difficulty in solving the equations of equilibrium, material and compatibility for such complex structures.

It appears that Mark (1982) was the first to make an elastic analysis of a Gothic building. In the 1960s, he used the photoelastic methods to obtain the stress distributions on a plane section of some French cathedrals. In the 1970s and 1980s, the elastic analysis was applied using the Finite Element Method (FEM), mainly through the work and influence of Mark. The problem is that the structure is modelled as a continuum, with known elastic properties, and precise information about boundary conditions. As this is not the case with masonry buildings, FEM elastic analysis of masonry architecture is a purely academic exercise.

Some packages allow simulating a unilateral material, incapable of resisting tension. There are several ways to do so but, in general, it is an iterative process. A first elastic analysis

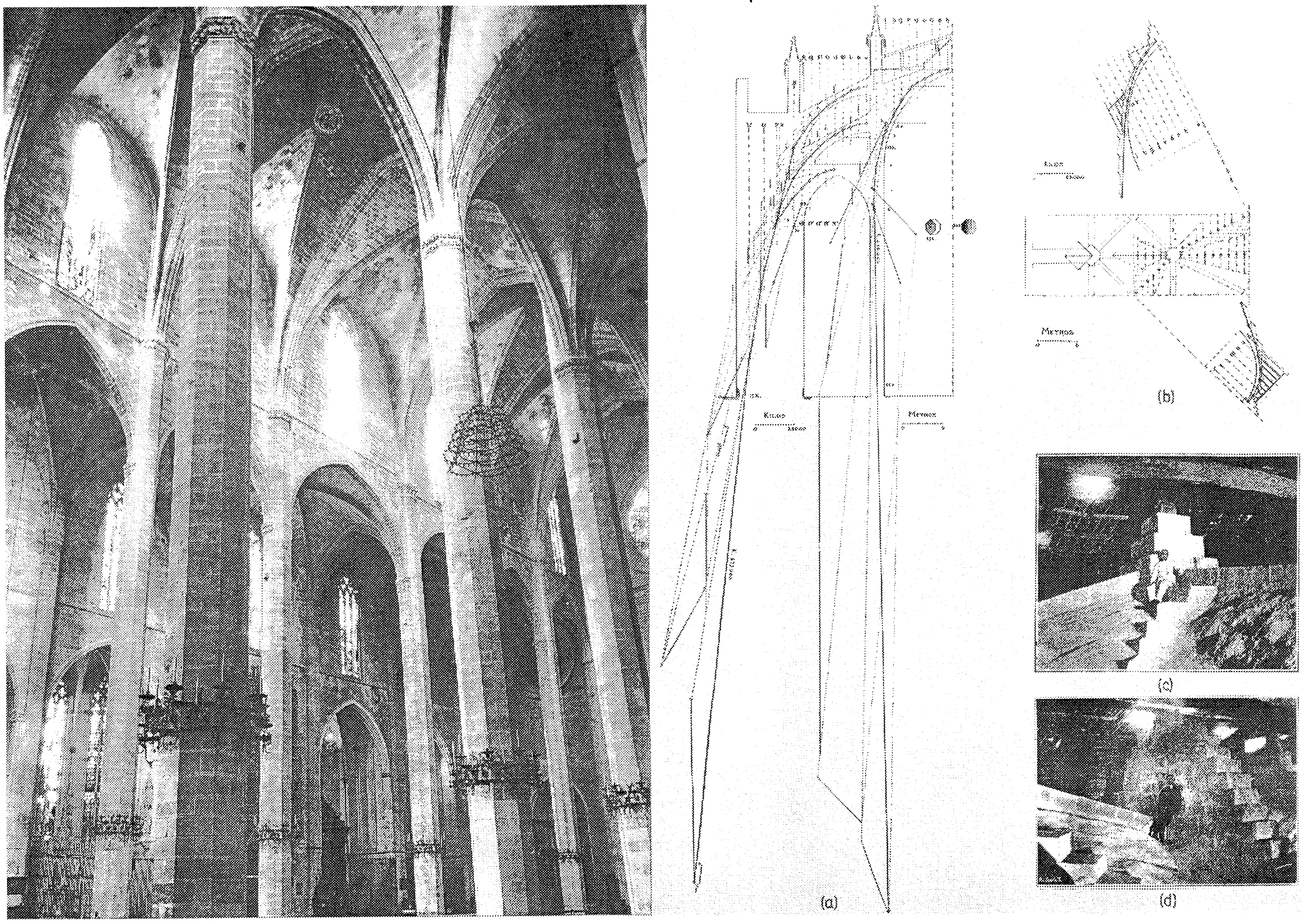


Figure 23: *Interior view of the slender columns of the cathedral of Palma de Mallorca: (a)-(b) equilibrium analysis of the cathedral of Palma de Mallorca by Rubió y Bellver (1912); (c)-(d) weights to equilibrate the slender pillars of the main nave.*

is made and the nodes where tension appears are broken, forming lines of discontinuity. The new structure is analyzed again, after a few iterations, the process converges, and a compressive solution is found. This is, of course, much better than the usual linear elastic analysis. Looking again at Figure 1 at the beginning, one wonders which kind of FEM model may be made, which considers the essential discontinuity and irregularity of the actual construction. Besides, the final state is dependant on the history of loads and movements suffered by the structure. As a result, these programs are only useful if the analyst is capable of interpreting the results and redirect the analysis if necessary.

For example, Barthel (1993) made an excellent study of Gothic cross vaults, where all the aspects have been considered and including a historical review of the methods employed. The book gathers, also, a lot of information on measurements and damages of actual buildings, a literature which is difficult to obtain. In one part of the work, Barthel uses non-linear FEM analysis to try to predict the cracks in different kind of vaults produced by certain yielding of the abutments. Being an experienced engineer, he is able to understand and interpret the results, and the study affords essential data for the knowledge of cross vault behaviour.

However, one of the results of the analysis was that, even a tiny movement, 0.5-2mm for a span of 10m (i.e., 1/20,000 - 1/5000 of the span!), produced cracking, a drastic change of internal stress state and of the value of the horizontal thrust which drops, depending on the form of the vault, up to 72% (see table in Figure 25). It is evident that a tool with such degree of uncertainty for such small movements, impossible to measure in a real building, is unusable. Then, one result of Barthel's study is that it demonstrates the inapplicability of FEM analysis to masonry vaults and buildings.

The same occurs with Discrete Element Methods (DEM), which are maybe the last computational approaches applied to masonry structures. In these methods, masonry is divided in block elements in contact. Nevertheless, again it is not possible to relate the drastic assumptions needed to run the program and the real building. As Heyman (1999) has pointed out, the set of equations of equilibrium, material and compatibility is extremely sensitive to very small variations on the boundary conditions.

In both cases, insofar as the program is using the equilibrium equations the results could be used and interpreted by an experienced engineer. Anyway, a pure equilibrium analysis will be required to check the results. Being this last analysis, as we shall see in the next paragraph, enough to check the safety of the structure, the use of these kinds of programs can only add "noise" to the analysis.

However, as it occurred with the elastic analysis of masonry arches at the end of the 19[th] Century, there is no way to stop the

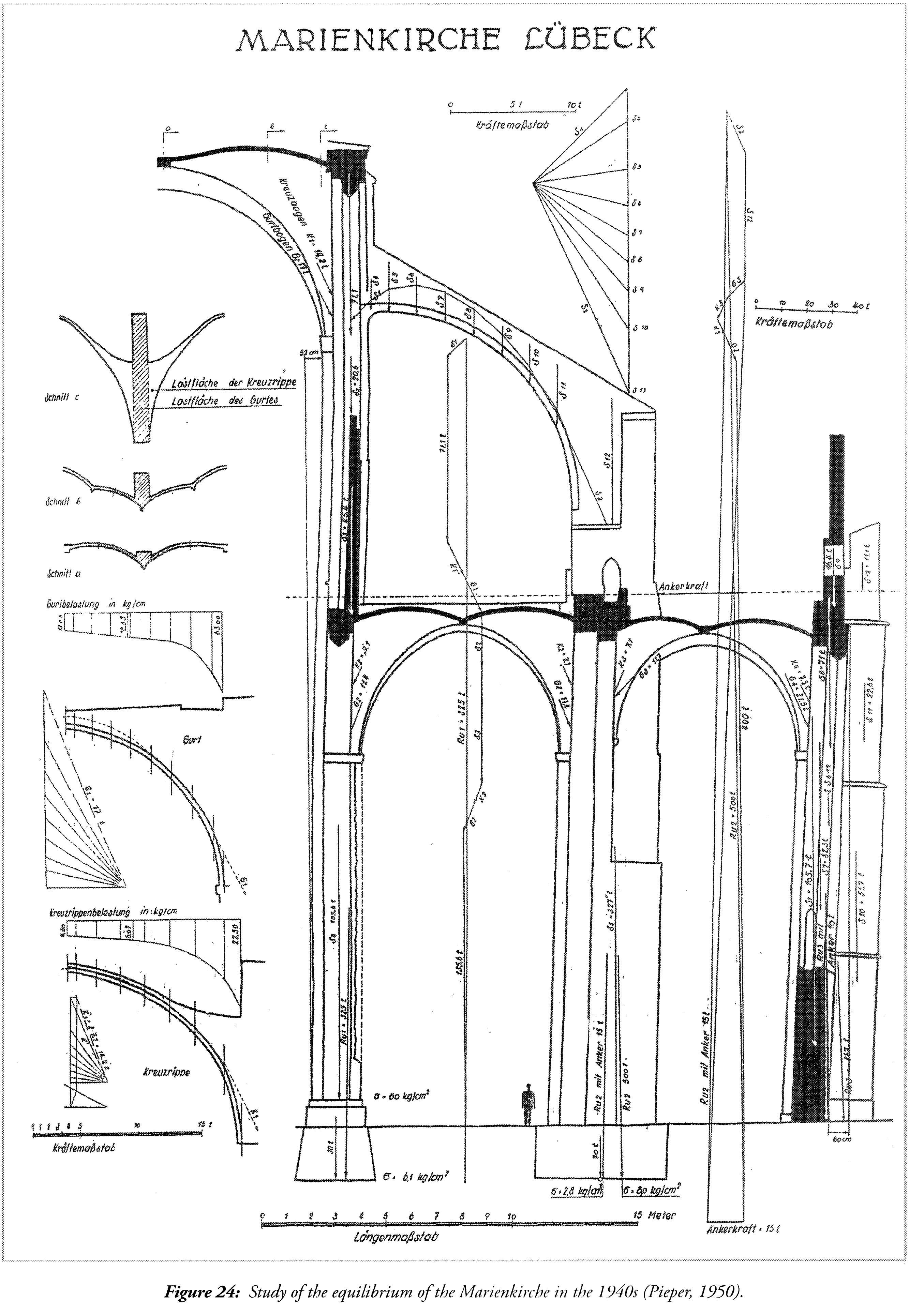
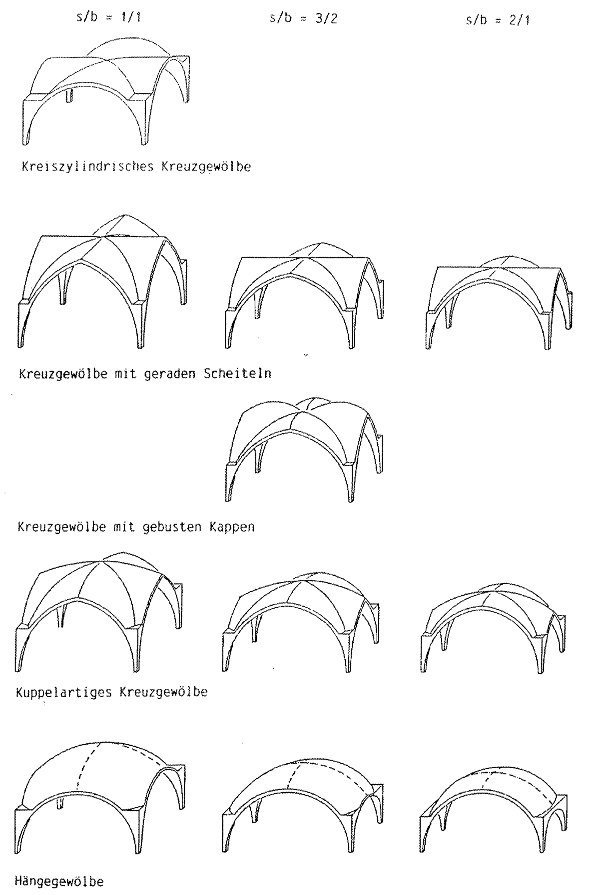


Figure 24: Study of the equilibrium of the Marienkirche in the 1940s (Pieper, 1950).



	s/b	v	$V/(g\frac{s}{2}\frac{b}{2})$	$H/(g\frac{s}{2}\frac{b}{2})$	h/f_k
Kreiszyl.	1/1	0,0	1,07	1,03	0,49
Kreuzgew.	1/1	2,0	1,07	0,77	0,34
Kreuzgew.	3/2	0,0	1,19	0,83	0,44
mit geraden	3/2	1,0	1,19	0,61	0,29
Scheiteln	1/1	0,0	1,17	0,73	0,45
	2/1	0,0	1,24	0,91	0,44
Gebustes	3/2	0,0	1,41	0,76	0,29
Kreuzgew.	3/2	0,5	1,41	0,73	0,28
Kuppel-	3/2	0,0	1,11	0,89	0,46
artiges	3/2	2,0	1,11	0,64	0,27
Kreuz-	1/1	0,0	1,13	0,78	0,44
gewölbe	2/1	0,0	1,12	0,94	0,48
Hänge-	3/2	0,0	1,14	0,83	0,34
gewölbe	3/2	2,0	1,14	0,68	0,27
	1/1	0,0	1,19	0,70	0,33
	2/1	0,0	0,82	0,89	0,39

s große Spannweite
b kleine Spannweite
f_k Stichhöhe des Kreuzbogens
g Gewicht pro Flächeneinheit
v Auflagerverschiebung in mm am berechneten System
V Vertikalkraft eines Viertels
H Horizontalkraft eines Viertels
h Höhe der Horizontalkraft über dem Kämpfer

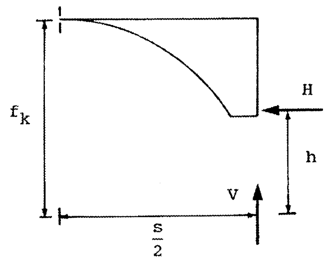


***Figure 25:** Barthel's (1993) analysis of cross vaults:*
(a) main types of cross vaults; (b) results of linear and non-linear FEM analysis of cross vaults.

wave that is sweeping common sense and critical observation of the buildings from the field of the analysis of historical constructions. The reader, consulting any congress proceedings or journals devoted to the topic, will see the diffusion of an approach, which, in many cases, goes against the very nature of the structures under study.

Limit Analysis of Masonry Structures

The Theory of the Limit Analysis of Structures, which was first formulated for steel frames, may be applied to masonry structures, if the masonry material fulfils certain conditions. In 1953, Drucker suggested to Kooharian (personal communication, cited in Kooharian, 1953) to apply limit analysis to the study of the collapse of voussoir arches; this was the first modern paper on the matter. In 1959, Prager discussed the conditions of the material of the voussoirs to apply rigorously the methods of limit analysis and draw the corresponding yield surface. However, it was Heyman in his milestone paper of 1966 who first exposed and discussed in a rigorous and general way the theory of Limit Analysis applied not only to voussoir arches, but also to any masonry structure. Since then, Heyman has contributed numerous papers (see reference list) that have illuminated the application to the theory to different masonry elements (arches, domes, fan vaults, cross vaults, towers, spires) and the way to asses the safety of these elements or of whole masonry structures. His contribution has been so fundamental that it will be difficult to imagine the state of the art now without his work. His papers have been compiled twice: first in a Spanish translation in 1995 (Heyman, 1995b) and then, as a reprint in English, in 1996. He outlined the basic theory of masonry structures in a book published in 1995. In what follows, we will summarize the main aspects of the theory, emphasizing how Heyman's approach solves the recurrent problems of the old theory of arches and vaults (a complete bibliography of Heyman's contributions may be found in Huerta, 2005c: xvii-xxiv).

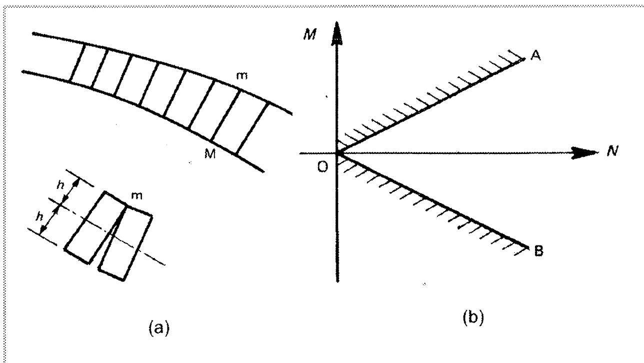
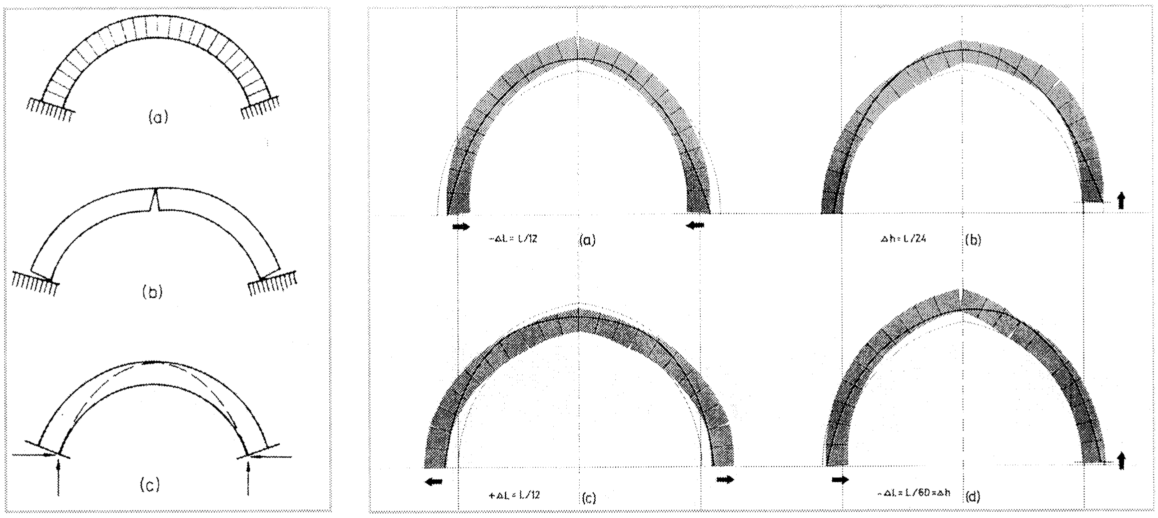


***Figure 26:** Yield surface:*
(a) forming of a hinge between two voussoirs;
(b) yield surface of rigid-unilateral masonry (Heyman, 1995a).



(a) (b)

***Figure 27:** A masonry arch over a centering:*
(a) cracking of a masonry arch after decentring (Heyman, 1995a);
(b) different patterns of cracks due to the movement of the springings (Huerta, 2005a).

The Material Masonry

To include the theory of masonry structures within the frame of Limit Analysis, the material masonry has to fulfil three conditions, which Heyman called the Principles of the Limit Analysis of Masonry structures: (1) masonry has an infinite compressive strength, (2) masonry has zero tensile strength, and (3) sliding between the blocks that form the masonry is impossible.

The first statement is unsafe as, obviously, there is no material of infinite strength. However, it is a fact that the stress levels in even the greatest masonry constructions are one or two order of magnitude below the crushing strength of the stone; it is a reasonable assumption, that can be checked at the end of the analysis. The second statement is safe, because the mortar between the stones had some adherence. Finally, the third statement is related with the high friction coefficients of masonry (0.6-0.7, corresponding to angles of friction between 35° and 39°): friction produced by the compressive stresses precludes the displacement by sliding. Again, this condition may be checked.

If the material masonry fulfils these conditions in a given section, the resultant of the acting stresses must be a compressive force contained within the section. When this stress resultant (the thrust between the two adjacent voussoirs) acts at the border, a hinge forms. This leads to a yield surface formed by two straight lines (Figure 26).

Masonry Arches: Cracks and the True Line of Thrust

We have seen that the engineers of the 19th Century considered that to know the position of the true line of thrust was fundamental and the main argument in favour of elastic analysis was precisely that permitted to fix the position of the line of thrust. Let us discuss the problem again under the frame of limit analysis.

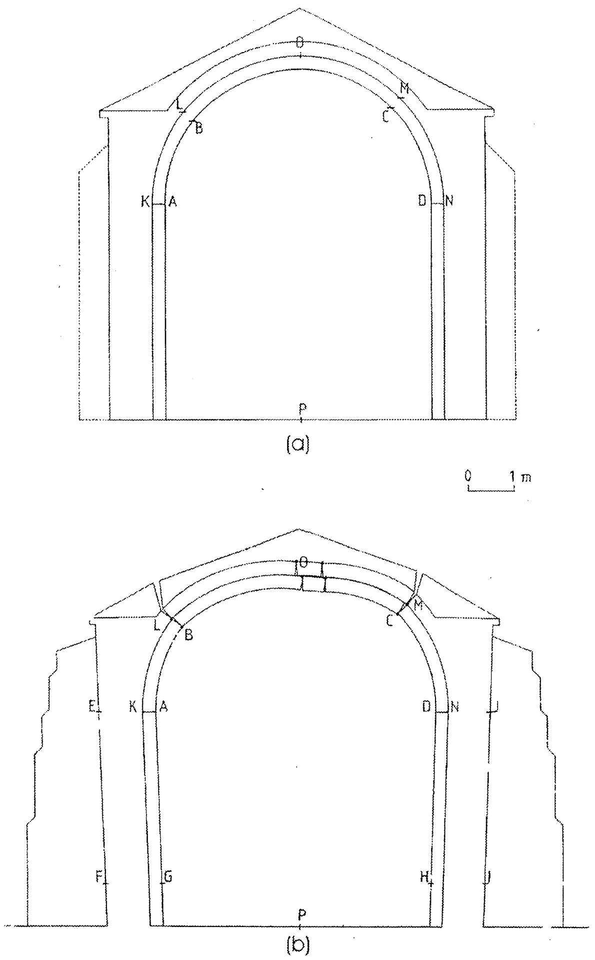
Consider a masonry arch over a centering (Figure 27a). After decentring, the arch begins to thrust against the abutments. Real abutments are not rigid and will yield a certain amount. The span increases and the arch must accommodate this change in geometry. The question is in what way could an arch (made of the rigid-unilateral material described above) do this? The answer, easily observed in many bridges and in tests with arch models, is by forming cracks that allow the movement: a crack opens at the keystone (downwards) and two other cracks open at the abutments (upwards).

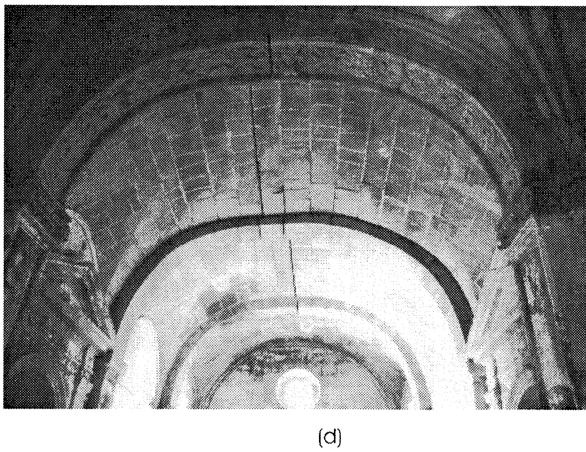
The arch becomes three-hinged and a unique line of thrust is possible. However, the movement may be asymmetrical; perhaps the right abutment, besides yielding horizontally, yields vertically. To every possible movement corresponds a certain cracking, and cracks open and close to permit the arch to respond to this aggression of the environment. This may be observed using models and even simple "plane" cardboard models give very good results (Figure 27b).

Therefore, cracks are not dangerous. The capacity of the structure to respond to the aggressions of the environment resides precisely in the possibility of cracking, which is due to the material properties: infinite compressive strength, no resistance to tension, and impossibility of sliding.

The cracking determines the position of the line of thrust. As the cracking varies, the line moves abruptly from one position to another (i.e., the internal forces change completely, as seen in Figure 27a). In the model, the movements are very large but even small movements, impossible to appreciate by inspection, have the same effect. It is obviously impossible to know or predict these kinds of perturbations. In fact, it is essentially impossible to know what is the actual line of thrust, i.e., in what state is the arch. However, there are two extreme positions of the line of thrust, which correspond to the minimum thrust and to the maximum thrust, as illustrated earlier in Figure 9a.

The cracks function like hinges and it is precisely the material properties cited above which allow hinge formation. This concept of "hinge" formation is crucial to the understanding of masonry structures. In particular, deformations are not "elastic"





		Distance (m)	Lean (m)	α (°)
walls	EF	3.77	0.135	2.05
	AG	3.77	0.095	1.45
	DH	3.77	0.100	1.52
	IJ	3.77	0.140	2.12

Table 1: Inside and outside inclination of the walls (α = inclination outwards)

		Original distance +	Increase =	Act. dist.
arch	AD	5.95	0.255	6.205
	BC	4.40	0.354	4.754
vault	KN	6.50	0.255	6.755
	LM	4.83	0.362	5.192
	OP	8.00	-0.325	7.675

Table 2. Original and actual distances (m)

(c)

Figure 28: *Barrel vault with gross deformations:*
(a) original state with insufficient buttresses; (b) actual state.
The drawing represents the real state without amplification of movements (Huerta & López- Manzanares, 1997).

in any sense; they are the result of the division of the structure in ascertain number of parts which, connected through the hinges, allow certain movements. In Figure 28, the original semicircular barrel vault, built with insufficient buttressing, cracked and the walls leaned until new massive buttresses were added. The drawing represents the actual state, without amplification of the movements (increase of circa 300mm of the original span of 6.5m). Cracks are not dangerous, but large unrestricted displacements of the abutments can lead to the catastrophic collapse of the structure.

Collapse of Arches

As the material has infinite strength, the collapse must be produced by the formation of cinematically admissible collapse mechanism (see Figure 4b earlier). When the line of thrust touches the limit of the masonry, a "hinge" forms allowing rotation. Three hinges make the arch statically determinate. One more hinge, however, may convert the arch in a four-bar cinematically admissible mechanism of collapse. Therefore, an increase of the load, which will lead to the formation of four alternate hinges, will lead to collapse without crushing of the material. This can occur in a stable arch with the addition of load that deforms sufficiently the line of thrust. Again, the hanging chain analogy makes the process clear, as illustrated in Figure 29.

Limit Analysis of Masonry Arches: The Fundamental Theorems

If we can draw a line of thrust within the arch, we know that this arch will have at least one possibility to stand for the given load conditions. However, does this mean that the arch will stand? Will it not be possible to find a way for the arch to collapse? Could a small, unforeseen movement cause cracking, lead to collapse?

The solution of the problem came only in the 20th Century with the theory of Limit Analysis and the demonstration of the fundamental theorems. There is insufficient space here to explain the origins and development of Limit Analysis, and the reader is directed to the books and articles of Heyman cited at the end of this paper. In particular, the Safe Theorem (or Lower Bound Theorem, or Static Theorem) states that if it is possible to find an internal system of forces in equilibrium with the loads which does not violate the yield condition of the material, the structure will not collapse and it is "safe". In the case of the masonry arch, any line of thrust for the given loads will satisfy the equilibrium conditions. The material requirements have been cited above, the main requirement being that the material must act in compression, i.e., in every section, the thrust (stress resultant) must be contained within the masonry. Therefore, if it is possible to draw a line of thrust (equilibrium) within the

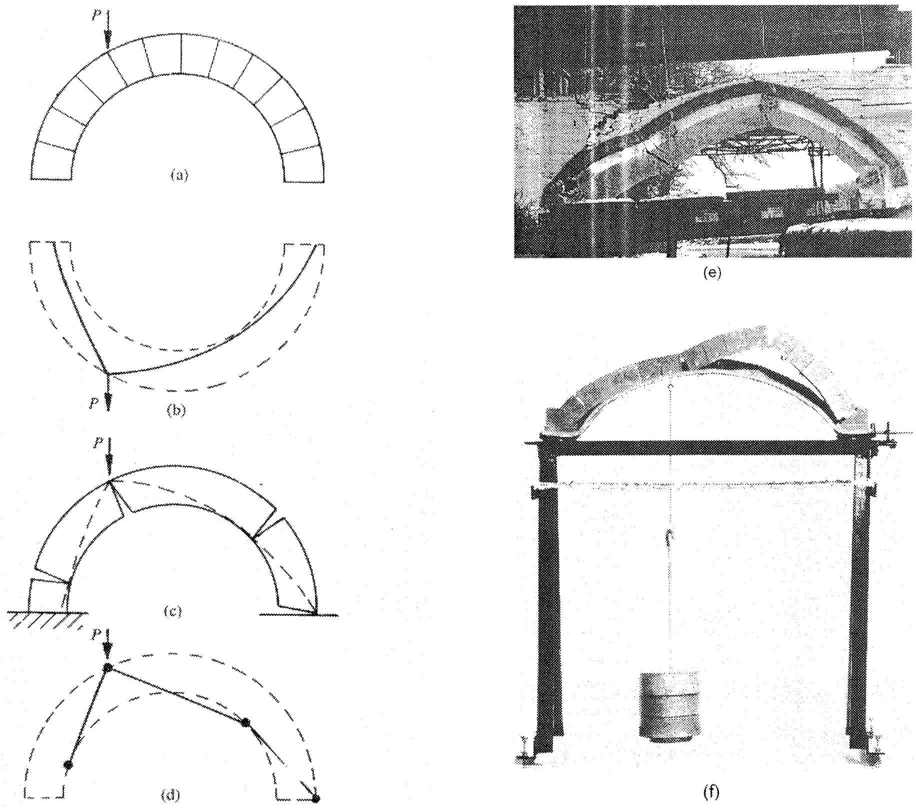


Figure 29: *Collapse of arches:*
(a)-(d) collapse of a semicircular masonry arch under a point load (Heyman, 1995a);
(e) collapse experiment on a real bridge (Page, 1993); (f) collapse of a model arch (Cowan, 1977).

arch (yield condition) this is an absolute proof that the arch is stable and that collapse will not occur for the given loading.

No affirmations are made about boundary conditions. The arch will crack as in Figure 27b responding to movements of the abutments, the line of thrust will move markedly finding new equilibrium states, but it will never go out of the masonry and it will never form a sufficient number of hinges to convert the arch in a mechanism. (Of course, if the structure is severely distorted, as in Figure 28, the calculations must be made with reference to the actual distorted geometry.)

Therefore, the Safe Theorem of Limit Analysis solves the problem of finding the actual line of thrust. It is impossible to know the actual line of thrust, but this is unimportant, as we can calculate the safety of the structure without making assumptions about its actual state.

The Safety of Masonry Arches

The fundamental theorems permit the calculation of the safety of masonry arches. Heyman (1969, 1982) proposed a geometrical factor of safety obtained comparing the geometry of the actual arch with that of the "limit arch" which will just support the loads. With reference to Figure 30, it is evident that by the Safe Theorem, the arch in Figure 30a will be safe; a possible line of thrust is comfortably within the masonry. Now, if we diminish the thickness of the arch at a certain value it will be possible to draw only one line of thrust contained within the arch. The line touches (due to the symmetry) in five points. We have then five hinges; the arch is in unstable

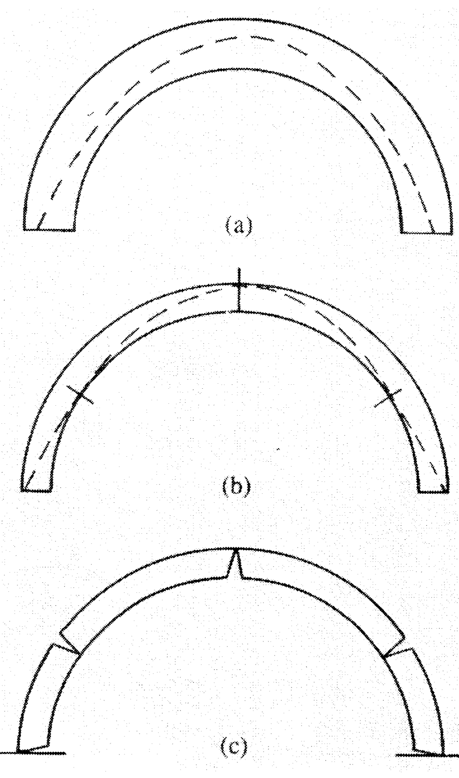


Figure 30: *Semicircular arch:*
(a) stable;
(b) of limit thickness (Heyman, 1995a).

equilibrium and would collapse. We can establish the safety of the original arch comparing its thickness with that of the limit arch. If the actual arch has double thickness, the geometrical factor will be two, and so on. In the case of a bridge, the limit arch for the worst position of the load should be found. The study of existing arches and bridges seems to recommend a value of two for the worst load condition (Heyman 1969, 1982).

To obtain the exact value of the geometrical factor of safety can involve heavy calculations. However, to obtain a lower bound may be very easy. For example, to show that for a certain arch under certain loads the geometrical factor of safety is equal or greater than two, it will suffice to draw a line of thrust within the middle-half of the arch. For a geometrical factor of three, within the middle-third, etc. The Safe Theorem confirms rigorously the intuition of many arch theoreticians of the last three centuries, from Gregory in the 17th, Couplet 18th, Rankine 19th, to Swain in the 20th Century.

Of course, an elastic analysis, which leads to a line of thrust within the arch, is a completely safe procedure. The only difference between the elastic analyst and the plastic or limit analyst, is that the first thinks that this line is the "actual" line and the second knows that it is impossible to know the actual line of thrust, and that the elastic line is one of the infinite possible in equilibrium with the loads contained within the arch.

The Analysis of Spatial Masonry Vaults: Heyman's Equilibrium Approach

One of the most important results of the Safe Theorem is that it permits an "equilibrium approach" to the analysis of structures. The first to note explicitly the equilibrium approach and its importance in the understanding of structural analysis and safety was Heyman: he mentioned it for the first time in 1967 (Heyman, 1967b) and then, in a more general way in 1969 (Baker & Heyman, 1969), and afterwards, many of his publications stressed the paramount importance of this corollary.

The task of the analyst is not to find the actual equilibrium state, but to find reasonable states of equilibrium for the structure under study. In fact, this has been the approach of all the great architects and engineers. It was implicit in the "geometrical design" of the master builders. It was explicit in the work of Maillart, Torroja, Nervi, Candela and Gaudí, to cite only a few great engineers and architects.

The equilibrium approach permits the analysis of complex vaults with reference to the arch theory already exposed. The technique consists of imagining the vault divided into a series of arches and then looking for a line of thrust inside each one of these arches. If this is possible, we have found a possible equilibrium solution in compression and, by the Safe Theorem, the structure is safe. We will review briefly some fundamental types.

Dome Analysis: Cracks and Equilibrium

The Safe Theorem justifies, also, the slicing technique. This method is simply a way to find easily a possible equilibrium state within a spatial structure, one between the infinitely many possible. A dome can be imagined as composed of a series of arches obtained by slicing the dome with meridian planes. Every two "orange slices" form an arch. If it is possible to draw a line of thrust within this arch, then we have found a possible equilibrium state in compression and the dome is safe; it will not collapse. This application of the Safe Theorem validates all the historical analysis using the "slicing technique" which we have studied in the first part of this paper.

Collapse analysis of domes is much more complicated: to demonstrate that a structure will collapse a cinematically admissible collapse mechanism must be found and this is not always easy in spatial vaults. The collapse studies of Kobell (1855) and Beckett (1871) in the second half of the 19th

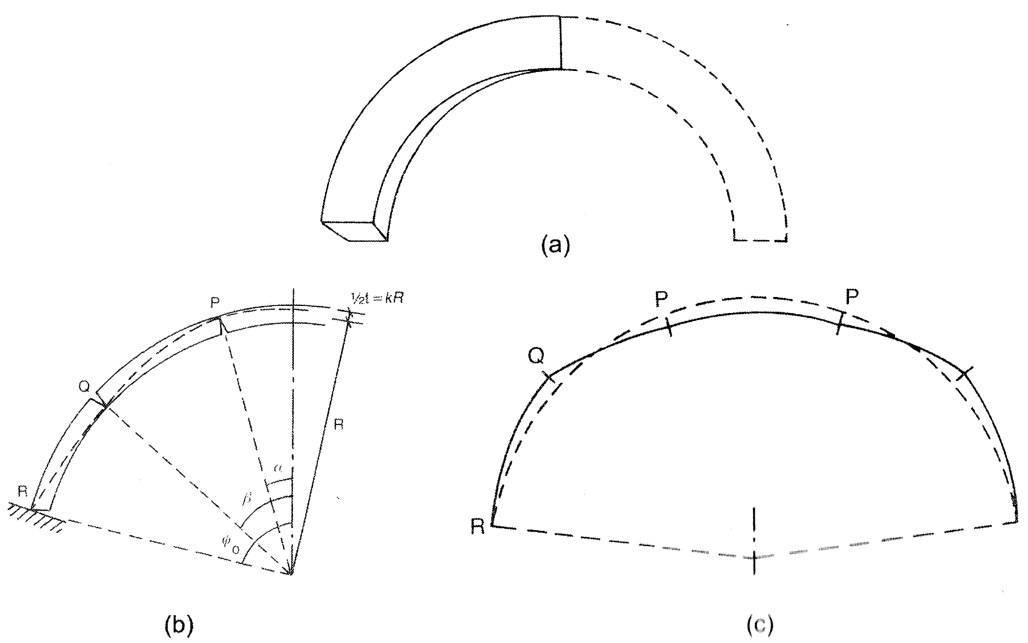


Figure 31: *Collapse of a semi-spherical masonry dome (Heyman 1988).*

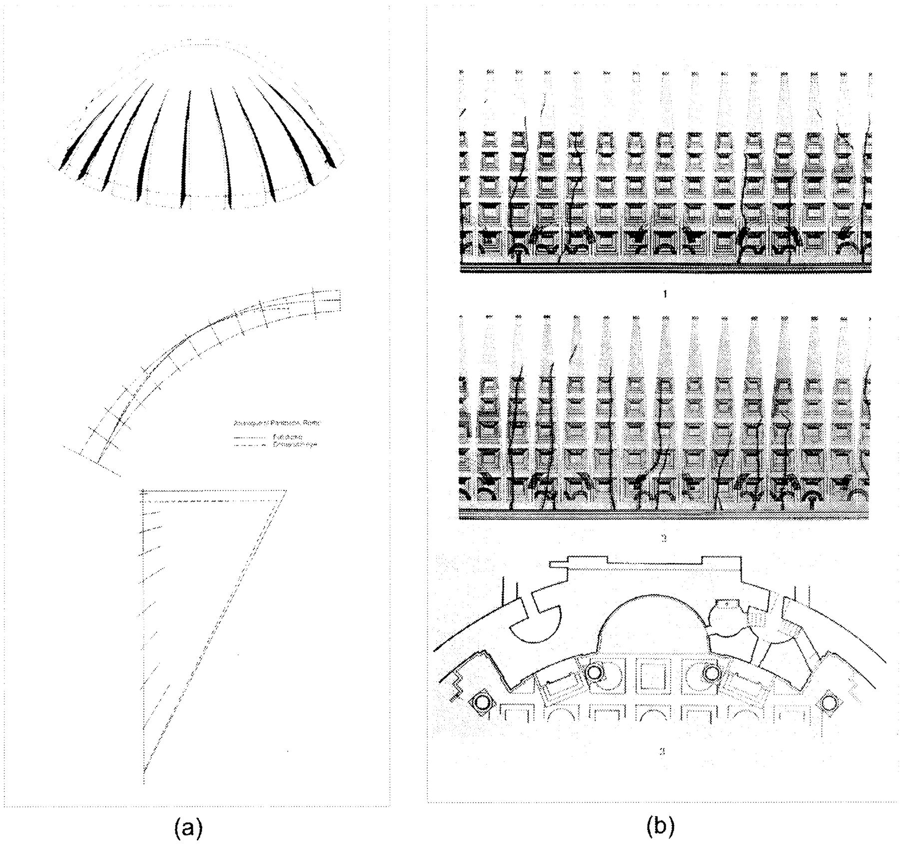


Figure 32: *Dome cracking:*
(a) typical cracking of a dome; (b) lines of minimum thrust in the Pantheon (Heyman, 1988);
(c) cracks in the dome of the Pantheon (Terenzio, 1933).

Century, for example, placed incorrectly the hinge at the crown. The first rigorous and correct study of the collapse of masonry domes was made by Heyman (1967a, 1977; see also Oppenheim, Gunaratnam & Allen, 1989).

As for the cracks, the typical pattern to be found in most cases is that of meridian cracks. This is produced by a slight yielding of the buttress system (of the tambour wall in most cases). We have seen that the *tre matemattici* were aware in 1742 of this fact, and this interpretation appears in some studies of dome cracks in the 19th Century.

The dome thrust outwards and the masonry of the abutment system gives way. Meridian cracks form inevitably, as seen in Figure 32a. The line (surface of thrust) moves upwards looking for the minimum thrust position. If the dome has no lantern, the line does not touch the extrados at the crown, but at some distance. The upper part, then, remain uncracked, a small dome-cap supported by a series of radial arches. The study of the line of thrust permits to calculate the theoretical extension of the cracks, and Heyman has done this for the dome of the Roman Pantheon, showing a good correlation with the cracks observed by Terenzio (1933) during the restoration of the 1930s (Figure 32b and 32c).

Gothic Vaults

The Safe Theorem also justifies the slicing technique for the analysis of Gothic vaults. Again, all the analysis described in the first part of this paper (Planat, Mohrmann and Pieper) are completely safe, and Heyman has revived this approach (1966,

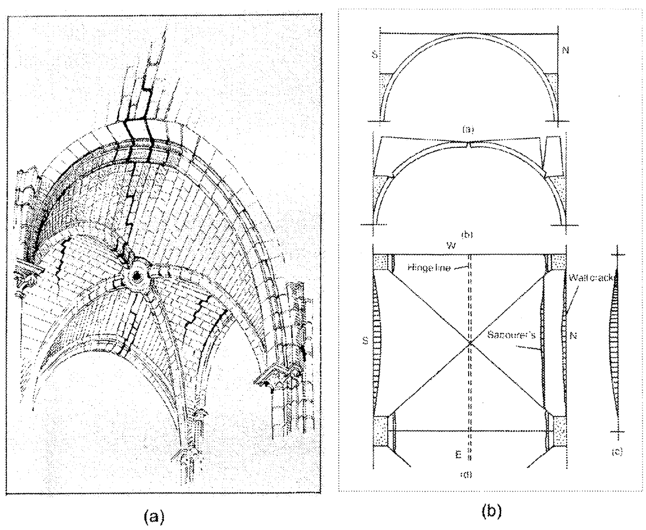


Figure 33: *Cracks in a Gothic quadripartite vault:*
(a) Abraham's description (1934);
(b) Heyman's interpretation of their origin (Heyman, 1983).

1977). In addition, the use of Mohrmann's table for the fast computing of the thrust is within the frame of the theory. Of course, different cuttings will give different values of the vault thrust, but if for any of them a system of lines of thrust can transmit the loads with a sufficient geometrical safety, then there is no doubt that the vault will not collapse and has at least the calculated safety.

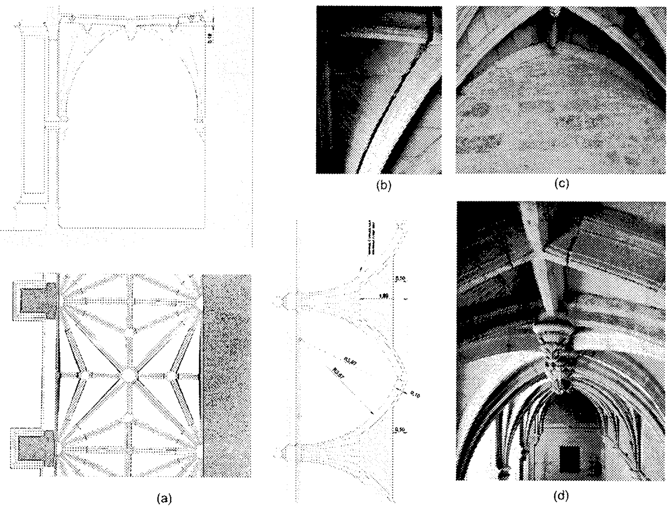


Figure 34: *Cracks in a real vault, the Monastery of Celanova (Spain): (a) plan (indicating the main cracks) and sections of the vaults, showing deformations and height of the fill over the extrados; b) Sabouret's crack; (c) crack at the wall; (d) hinge line at the crown (Huerta, 2007).*

With reference to the cracks in Gothic vaults, the same observations apply as in domes. Cracks should be interpreted as dividing the structures in a certain number of blocks, which permits the movement imposed by the environment. As we have seen, architects and engineers were well aware of the existence of cracks. Figure 33a shows Abraham's (1934) drawing of typical cracks and Figure 33b Heyman's (1983) interpretation. The longitudinal barrel cracks in the typical archway; the transverse barrel must break to accompany the movement. The ribs will follow the general movement (Heyman, 1983). The cracking does not alter the safety of the vault, if the movements are small, the equilibrium equations written before and after the movement are the same. Of course, actual cracks are heavily dependant in the actual geometry and construction of the vault. In Figure 34, the webs of the small vaults at the cloister of the Monasterio de Celanova are formed by single stones between the ribs. Sabouret and wall cracks form now over the ribs, the webs separating and functioning like arches supported on the high masonry fill. The hinge cracks adopt the usual pattern. Again, the cracks do not affect the global safety of the vault.

Masonry Buildings: Equilibrium, Cracks and Safety

The Safe Theorem validates the equilibrium analysis in compression made by Planat, Mohrmann and Rubió, which have been discussed above. It is important not to forget the objectives of the analysis. Usually in the study of historic buildings, there are two main objectives: (1) to understand the way the structure behaves, and (2) to understand the origin and significance of the cracks, if they are visible. Only with a good comprehension of both aspects can the engineer or architect judge the situation and decide what to do (if there is anything to do at all). The first task implies the study of possible states of equilibrium. The second to imagine what kind of movements

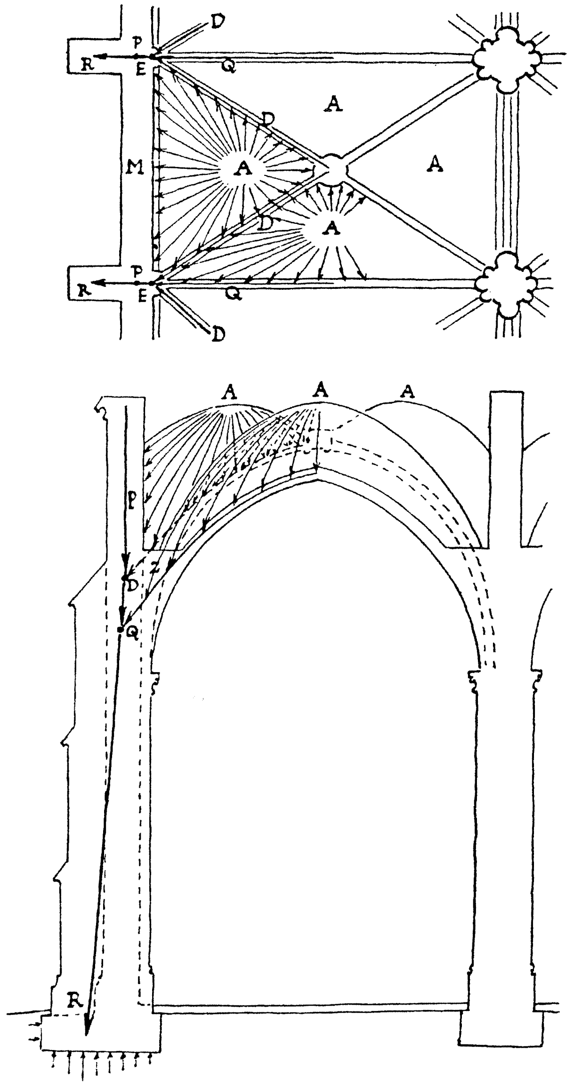


Figure 35: *Possible state of equilibrium in compression in a masonry vaulted building (Rave, 1939-40).*

have given origin to the observed pattern of cracks.

The study of equilibrium can be best made first "identifying" the elements that compose the structure (in fact, the decision of what is structure and what is not is the very first part of analysis). In the Gothic cathedrals, the structure and its elements are very apparent; in Romanesque or Byzantine churches, it is not so evident. It should be emphasised that this previous analysis has a paramount importance: an inadequate identification of the structure and its elements will be misleading. Fortunately, the number of basic types of historic masonry structures and their elements is limited and Heyman has already studied the most basic types.

The second task is, in general, more difficult. The analyst should bear in mind the typical cracking patterns for the different types of arches and vaults and their relation to the abutment movements, and imagine by analogy, what kind of movements could have originated the actual pattern of cracks in the actual building. Complicated patterns produced by combined movements will demand the analyst experience and insight. As has been mentioned before, cracks in themselves do not affect the safety of a masonry structure; they are, simply, a manifestation of the nature of the material. By cracking, the masonry structure can adapt itself to little (unavoidable and essentially unknowable) movements of the environment (e.g., soil settlements). In Figure 35, the lines and surfaces of thrust represent a possible static state of equilibrium (in compression). A small movement of one of the supports (for example a vertical settlement) will inevitably produce a visible cracking in the vaults. If the movement is asymmetrical the pattern will be irregular and more difficult to interpret. However if the movement is small (say 1/100 of the span, or 100 mm for a 10 m span), the same state of equilibrium will be contained within the distorted geometry; in fact, 1/100th of the span is more or less the thickness of the lines of the drawing. Therefore, the cracking, however visually alarming for a modern architect or engineer is harmless. Besides, this pattern may change with another variation of the boundary conditions. The apparent precision afforded by some computer packages that may reproduce, perhaps, the cracking at a certain moment has no sense, as there is no way to predict future movements. Nevertheless, the Safe Theorem assures us, beyond any doubt, that the structure will remain safe for any "small" movements in the future.

Sometimes the equilibrium approach is considered as a synonym of graphical statics. Graphical statics is one possible *tool* and the equilibrium approach is the *theoretical frame* rigorously deduced from the Fundamental Theorems of Limit Analysis. The equilibrium equations may be handled graphically or analytically, and this makes no difference. Some problems are treated better graphically: the analyst see the working of the forces and the form of the structure and being the stability of masonry architecture a matter of geometry, it is sensible to employ geometrical methods. However, some problems require a degree of precision, which recommends an analytical solution. In other cases, the trials are more economically made writing directly the equilibrium equations and studying the variation of the main parameters.

Computer programs may be a great help, of course. The first to implement a computer model for an arch made of rigid voussoirs was Livesley (1978) and since there have been quite a number of theoretical contributions (see, for example, Sinopoli, 1998). However, only few computer packages are available to help the analyst in the study of the equilibrium of arches, vaults and buttresses. For masonry arch bridges, there are good computer programs (for example, *Archie-M* and *Ring*) which work within the assumptions of limit analysis. In the case of masonry buildings, the geometry is much more complex; it is more difficult to produce programs easy to use and which may be adapted to the building in question. O'Dwyer (1999) has proposed a funicular approach, which is the computer version of Gaudí's funicular models. Maunder (1995) has discussed the possibility to adapt FEM packages to the type of block equilibrium approach above discussed. Ochsendorf (2006) and his collaborators at MIT are now actively working in producing programs to study the equilibrium and safety of masonry structures and the results so far are very attractive and their employment will be very useful in the teaching of the behaviour of such structures (Block, Ciblac & Ochsendorf, 2006; Ochsendorf , 2006).

Any of the cited programs may help the analyst in the study of certain typical elements. However, for anyone involved in the analysis of historical structures, it is clear that every building, even within a certain studied type, presents different problems and there is not a unique way to approach the problem. Specialized monographs are needed, and a register of analysis of particular types, subsequently compared with observed damages will also improve considerably our capacity of analysis.

Conclusions

The rock on which the whole of structural theory is now seen to be provided by the Safe Theorem. J. Heyman, *The Science of Structural Engineering,* 1999, p 101.

The theory of masonry structures is based in the study of reasonable equilibrium states in compression, of sets of compressive stress resultants within the masonry in equilibrium with external loads. The approach is already present in the "old theory" developed during the 17th and 18th Centuries and applied successfully through all the 19th Century. As discussed, Heyman has shown how this approach may be incorporated within the frame of modern Limit Analysis. If masonry can be considered strong enough not to fail in compression, the weak tensile strength is ignored, and the geometry is such as to preclude sliding, then, the Fundamentals Theorems of Limit Analysis can be translated to masonry. Heyman has also shown, crucially, that the main corollary of the Safe Theorem is precisely the *equilibrium approach* that was used intuitively by all the great masonry builders. Indeed, it is embedded in the geometrical design rules of the old master builders and was used by the great engineers and architects of the 18th and 19th Centuries for the design of bridges and buildings. It follows directly from the Safe Theorem of Limit Analysis applied to masonry structures.

In considering only possible equilibrium solutions that respect the essential no-tension character of the material, the analyst is led to consider only the fundamental problems in question. No consideration is made of the changing and essentially unknowable boundary conditions. No affirmation is made about elastic properties of the masonry. The essentials of the complexity of the structure are now under consideration. The task is not easy; no computer program will give us a unique answer, but the problem presents itself with all its fascinating complexity and richness. Now the analyst is in the situation to ask relevant questions and give meaningful answers. It is not an amateur task; it needs long years of study, practice and reflection.

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